Supplemental Material

Optimal tax policy under heterogeneous environmental preferences

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Closed-form expressions

We first provide two technical lemmas that will be useful to obtain closed-form expressions for Φ and p_i^* and to derive comparative statics results on p_i^*.

Let $\mathcal{K}=\{1,...,K\}$, $\beta=\{\beta_1,...,\beta_K\}$ and $\mathbf{B}$ a $K\times K$ matrix such that $\begin{pmatrix} 1 & \beta_2 & \cdots & \beta_K \\ \beta_1 & 1 & \cdots & \beta_K \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1 & \beta_2 & \cdots & 1 \end{pmatrix}$.

Let $M_{ij}(\mathbf{B})$ be the i,j minor matrix of $\mathbf{B}$ (i.e. we remove its i^{th} row and j^{th} column). In particular, let $\mathbf{B}^{\mathcal{K}\setminus i}\equiv M_{ii}(\mathbf{B})$. Let $\mathbf{B}_{j,a}$ be the matrix $\mathbf{B}$ to which we have replaced all values of β_j by a. The matrix $\mathbf{B}_{j,a}^{\mathcal{K}\setminus i}$ combines the two modifications to $\mathbf{B}$.

Lemma 1. $\det(\mathbf{B})=\sum_{S\subseteq\mathcal{K}}(-1)^{|S|+1}\left(|S|-1\right)\prod\limits_{j\in S}\beta_j$.

Proof. Consider $\mathcal{K}=\{1,2\}$. Then, $\det(\mathbf{B})=1-\beta_1\beta_2$ which is of the desired form.

Suppose now that $\mathcal{K}=\{1,...,K\}$. We proceed by induction. Suppose that we have shown the result if $|\mathcal{K}|=K-1$.

We have that

$$\det(\mathbf{B})=\det(\mathbf{B}^{\mathcal{K}\setminus K})+\sum_{l=1}^{K-l}\beta_K(-1)^l\det\left(M_{K-l,K}(\mathbf{B})\right).$$

Notice that $M_{K-l,K}(\mathbf{B})$ contains a column of α_{K-l}. Replace it by a column of 1 and move the column in last position, moving all columns in position $K-l+1$ to $K-1$ by one position to the left, for a total of $l-1$ column inversions. Notice that this new matrix is $\mathbf{B}_{K-l,1}^{\mathcal{K}\setminus K}$. We thus have that $\det\left(\mathbf{B}_{K-l,1}^{\mathcal{K}\setminus K}\right)=(-1)^{l-1}\beta_{K-l}\det(M_{K-l,K}(\mathbf{B}))$. Thus, we have

$$\begin{aligned}\det(\mathbf{B}) &= \det(\mathbf{B}^{-K})-\beta_K\sum_{l=1}^{K-l}\beta_{K-l}\det\left(\mathbf{B}_{K-l,1}^{\mathcal{K}\setminus K}\right)\\ &= \sum_{S\subseteq\mathcal{K}\setminus K}(-1)^{|S|+1}\left(|S|-1\right)\prod_{j\in S}\beta_j\\ &\quad -\beta_K\sum_{l=1}^{K-l}\beta_{K-l}\sum_{S\subseteq\mathcal{K}^{K-l}}(-1)^{|S|+1}\left(|S|-1\right)\prod_{j\in S}\beta_j.\end{aligned}$$

where $\mathcal{K}^{K-l}=\mathcal{K}\cup\{K+1\}\setminus\{K,K-l\}$, $\beta_{K+1}=1$ and where the last equality is by the induction hypothesis.

Next, notice that if $\beta_i = 1$ for some $i \in \mathcal{K}$, then

$$\begin{aligned}\det(\mathbf{B}) &= \sum_{S \subseteq \mathcal{K}\setminus i} (-1)^{|S|+1} \left(|S| - 1 - |S|\right) \prod_{j \in S} \beta_j \\ &= \sum_{S \subseteq \mathcal{K}\setminus i} (-1)^{|S|} \prod_{j \in S} \beta_j.\end{aligned}$$

Thus, we have

$$\begin{aligned}\det(\mathbf{B}) &= \sum_{S \subseteq \mathcal{K}\setminus K} (-1)^{|S|+1} \left(|S| - 1\right) \prod_{j \in S} \beta_j \\ &\quad -\beta_K \sum_{l=1}^{K-l} \beta_{K-l} \sum_{S \subseteq \mathcal{K}\setminus\{K,K-l\}} (-1)^{|S|} \prod_{j \in S} \beta_j. \\ &= \sum_{S \subseteq \mathcal{K}\setminus K} (-1)^{|S|+1} \left(|S| - 1\right) \prod_{j \in S} \beta_j \\ &\quad + \sum_{S \subseteq \mathcal{K}\setminus K} (-1)^{|S|+1} |S| \, \beta_K \prod_{j \in S} \beta_j \\ &= \sum_{S \subseteq \mathcal{K}} (-1)^{|S|+1} \left(|S| - 1\right) \prod_{j \in S} \beta_j\end{aligned}$$

where the second equality follows from the following argument: fix $T \in \mathcal{K}\setminus K$. Then, $\prod_{j \in T} \beta_j$ appears in $\sum_{l=1}^{K-l} \beta_{K-l} \sum_{S \subseteq \mathcal{K}\setminus\{K,K-l\}} (-1)^{|S|} \prod_{j \in S} \beta_j$ whenever we select $l = j$ such that $K - j \in T$ and $S = T \setminus \{K - j\} \subseteq \mathcal{K} \setminus \{K, K - j\}$. Thus, $\prod_{j \in T} \beta_j$ appears exactly $|T|$ times. □

Lemma 2. *Suppose that* $\mathcal{K} = \{1, ..., K\}$, $\beta = \{\beta_1, ..., \beta_K\}$ *and* $\beta_j \in \left[0, \frac{1}{K-1}\right]$ *for all* $j \in \mathcal{K}$, *then* $det(\mathbf{B}) > 0$.

Proof. By Lemma 1, $\det(\mathbf{B}) = \sum_{S \subseteq \mathcal{K}} (-1)^{|S|+1} \left(|S| - 1\right) \prod_{j \in S} \beta_j$. Notice that

$$\sum_{S \subseteq K\setminus i} (-1)^{|S|+1} (|S|-1) \Pi_{j \in S} \beta_j = 1 - \sum_{\substack{S \subseteq \mathcal{K}\setminus i \\ |S|=2}} \Pi_{j \in S} \beta_j + \sum_{m=1}^{\frac{K-1}{2}} \left(\sum_{\substack{S \subseteq \mathcal{K}\setminus i \\ |S|=2m+1}} 2m \Pi_{j \in S} \beta_j - \sum_{\substack{S \subseteq \mathcal{K}\setminus i \\ |S|=2m+2}} (2m+1) \Pi_{j \in S} \beta_j \right).$$

We first show that $1 - \sum_{\substack{S \subseteq \mathcal{K}\setminus i \\ |S|=2}} \Pi_{j \in S} \beta_j \geq 0$. Notice that since $\beta_j \leq \frac{1}{K-1}$, $\beta_j \beta_{j'} \leq \frac{1}{(K-1)^2}$. We have $\frac{(K-1)!}{2!(K-3)!} = \frac{(K-1)(K-2)}{2}$ coalitions of size 2 in $K \setminus i$, thus $\sum_{\substack{S \subseteq \mathcal{K}\setminus i \\ |S|=2}} \Pi_{j \in S} \beta_j \leq \frac{1}{(K-1)^2} \frac{(K-1)(K-2)}{2} = \frac{(K-2)}{2(K-1)} < 1$.

We next show that $\sum_{\substack{S \subseteq \mathcal{K}\setminus i \\ |S|=2m+1}} 2m \Pi_{j \in S} \beta_j \geq \sum_{\substack{S \subseteq \mathcal{K}\setminus i \\ |S|=2m+2}} (2m+1) \Pi_{j \in S} \beta_j$ for all $m = 1, ..., \frac{K-1}{2}$. We want to use the fact that terms in the LHS contain a multiplication of $2m + 1$ different β_j, while on the RHS it is a multiplication of $2m+2$ different β_j. Take S such that $|S| = 2m+1$. Then $\Pi_{j\in S\cup l}\beta_j = \Pi_{j\in S}\beta_j * \beta_l \leq \Pi_{j\in S}\beta_j \frac{1}{K-1}$, so $\Pi_{j\in S\cup l}\beta_j (K-1) \leq \Pi_{j\in S}\beta_j$. In other words, a term that contains a multiplication of $2m+1$ different β_j can be use to cancel out $(K-1)$ terms containing a multiplication of $2m+2$ different β_j (provided that they differ by a single β_j).

We have $\frac{(K-1)!}{(2m+1)!(K-2m-2)!}$ coalitions of size $2m+1$, each of them having a coefficient of $2m$. We have $\frac{(K-1)!}{(2m+2)!(K-2m-3)!}$ coalitions of size $2m+2$, each of them having a coefficient of $2m+1$. Thus, to obtain our result,[1] we need

$$\begin{aligned} \frac{(K-1)!}{(2m+1)!(K-2m-2)!}2m &> \frac{(K-1)!}{(2m+2)!(K-2m-3)!}\frac{2m+1}{K-1} \\ (K-1)(2m+2)2m &> (K-2m-2)(2m+1) \end{aligned}$$

which is obviously verified, as $(K-1) > (K-2m-2)$ and $2m+2 > 2m+1$.

□

Corollary 1. $\det(\mathbf{B}) = \sum_{S\subseteq\mathcal{K}}(-1)^{|S|+1}(|S|-1)\prod_{j\in S}\beta_j > 0.$

Proof. Lemma 1 allows us to obtain $\det(\mathbf{B}) = \sum_{S\subseteq\mathcal{K}}(-1)^{|S|+1}(|S|-1)\prod_{j\in S}\beta_j$, and Lemma 2 to obtain that $\det(\mathbf{B}) > 0$. □

We next find the closed-form expression of $\sum_{j\in\mathcal{K}}(-1)^{j+i}M_{ij}(\mathbf{B})$.

Lemma 3. *For all $i \in \mathcal{K}$,*

$$\sum_{j\in\mathcal{K}}(-1)^{j+i}M_{ij}(\mathbf{B}) = \sum_{S\subseteq\mathcal{K}\setminus i}(-1)^{|S|+1}\left((K-|S|-1)\beta_i + |S|-1\right)\prod_{j\in S}\beta_j \equiv A_i.$$

Proof. For $j \neq i$, apply the following modifications to $M_{ij}(\mathbf{B})$:

i) The matrix contains a column of β_i, that we replace by a column of 1. The modified matrix will have a determinant that is $\beta_i \det(M_{ij}(\mathbf{B}))$.

ii) Move this column (which is in position i if $j > i$ and $i-1$ if $j < i$) to position i, moving all columns in between by one position towards position i. This is exactly $i-j-1$ column inversions if $i > j$ and $j-i-1$ column inversions if $i < j$. This change means that we have to multiply the determinant by $(-1)^{|i-j|-1}$.

[1] Formally, we also need to verify that we can pair terms on the LHS and RHS, with, in each pair, the term on the left containing one less β_j than on the right. This verification is straightforward but space-intensive and is left to the reader.

Notice that the modified matrix is exactly $\mathbf{B}_{j,1}^{\mathcal{K}\setminus i}$. Recall that $M_{ii}(\mathbf{B}) = \mathbf{B}^{\mathcal{K}\setminus i}$. Thus,

$$\begin{aligned}
\sum_{j\in\mathcal{K}}(-1)^{j+i}M_{ij}(\mathbf{B}) &= \det(\mathbf{B}^{\mathcal{K}\setminus i}) + \sum_{j\in\mathcal{K}\setminus i}(-1)^{j+i}(-1)^{|i-j|-1}\beta_i \det(\mathbf{B}_{j,1}^{\mathcal{K}\setminus i}) \\
&= \det(\mathbf{B}^{\mathcal{K}\setminus i}) - \beta_i \sum_{j\in\mathcal{K}\setminus i}\det(\mathbf{B}_{j,1}^{\mathcal{K}\setminus i}) \\
&= \sum_{S\subseteq\mathcal{K}\setminus i}(-1)^{|S|+1}(|S|-1)\prod_{l\in S}\beta_l \\
&\quad -\beta_i \sum_{j\in\mathcal{K}\setminus i}\sum_{S\subseteq\mathcal{K}^l}(-1)^{|S|+1}(|S|-1)\prod_{l\in S}\beta_l.
\end{aligned}$$

where $\mathcal{K}^l = \mathcal{K}\cup\{K+1\}\setminus\{i,j\}$ and $\beta_{K+1} = 1$. We can then simplify to

$$\begin{aligned}
\sum_{j\in\mathcal{K}}(-1)^{j+i}M_{ij}(\mathbf{B}) &= \sum_{S\subseteq\mathcal{K}\setminus i}(-1)^{|S|+1}(|S|-1)\prod_{l\in S}\beta_l & (1)\\
&\quad -\beta_i \sum_{j\in\mathcal{K}\setminus i}\sum_{S\subseteq\mathcal{K}\setminus\{i,j\}}(-1)^{|S|}\prod_{l\in S}\beta_l \\
&= \sum_{S\subseteq\mathcal{K}\setminus i}(-1)^{|S|+1}(|S|-1)\prod_{l\in S}\beta_l & (2)\\
&\quad -\beta_i \sum_{S\subseteq\mathcal{K}\setminus i}(-1)^{|S|}(K-|S|-1)\prod_{j\in S}\beta_j \\
&= \sum_{S\subseteq\mathcal{K}\setminus i}(-1)^{|S|+1}((K-|S|-1)\beta_i + (|S|-1))\prod_{j\in S}\beta_j \\
&= A_i.
\end{aligned}$$

where equation (2) comes from the fact that a coalition T appears in the second line of equation (1) whenever $i \notin T$ and $S = T$, so $K - |S| - 1$ times. □

We want to transform $\mathbf{BY} = \mathbf{N}$ into $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{N} = \frac{1}{det(\mathbf{B})}adj(\mathbf{B})\mathbf{N}$. For p_i^* we need to sum the i^{th} column of the adjoint matrix only, which is $\sum_{j\in\mathcal{K}}(-1)^{j+i}M_{ij}(\mathbf{B}) = A_i$.

Theorem 2. *$\Phi(n_1,...,n_K,N_1,...,N_K,\beta_1,...,\beta_K) = \left(\frac{1}{det(\mathbf{B})}\right)\sum_{j\in S}A_j(N_j - n_j)$ and $p_i^* = 1 - \frac{A_i}{det(\mathbf{B})}$ for all $i \in \mathcal{K}$.*

Proof. Follows from Corollary 1 and Lemma 3. □

Comparative statics

Notice that for all $i \in N$,

$$\begin{aligned}\det(\mathbf{B}) - A_i &= \sum_{S \subseteq K \setminus i} (-1)^{|S|+1} (1-K)\beta_i \prod_{j \in S} \beta_j \\ &= (K-1)\beta_i \sum_{S \subseteq K \setminus i} (-1)^{|S|} \prod_{j \in S} \beta_j.\end{aligned}$$

Given that $K > 1$ and that it is easy to verify that $\sum_{S \subseteq K \setminus i} (-1)^{|S|} \prod_{j \in S} \beta_j > 0$ for $\beta_j \in \left[0, \frac{1}{K-1}\right]$, we have that $\det(\mathbf{B}) - A_i > 0$ (except in the cases $K = 1$, and/or $\beta_i = 0$).

Theorem 3. *For all* $i \in \mathcal{K}$ *and* $\beta_i \neq 0$, $\frac{\partial p_i^*}{\partial \beta_i} > 0$.

Proof. We have that

$$\frac{\partial p_i^*}{\partial \beta_i} (\det(\mathbf{B}))^2 = \frac{\partial(\det(\mathbf{B}) - A_i)}{\partial \beta_i} \det(\mathbf{B}) - (\det(\mathbf{B}) - A_i) \frac{\partial \det(\mathbf{B})}{\partial \beta_i}$$

Note that

$$\frac{\partial(\det(\mathbf{B}) - A_i)}{\partial \beta_i} = \frac{(\det(\mathbf{B}) - A_i)}{\beta_i}.$$

Therefore:

$$\begin{aligned}\frac{\partial p_i^*}{\partial \beta_i} (\det(\mathbf{B}))^2 &= \frac{(\det(\mathbf{B}) - A_i)}{\beta_i} \det(\mathbf{B}) - (\det(\mathbf{B}) - A_i) \frac{\partial \det(\mathbf{B})}{\partial \beta_i} \\ &= (\det(\mathbf{B}) - A_i) \left(\frac{\det(\mathbf{B})}{\beta_i} - \frac{\partial \det(\mathbf{B})}{\partial \beta_i} \right). \\ &= (\det(\mathbf{B}) - A_i)\beta_i \left(\det(\mathbf{B}) - \beta_i \frac{\partial \det(\mathbf{B})}{\partial \beta_i} \right)\end{aligned}$$

We have that

$$\det(\mathbf{B}) - \beta_i \frac{\partial \det(\mathbf{B})}{\partial \beta_i} = \det(\mathbf{B_{i,0}}) > 0.$$

Thus,

$$\frac{\partial p_i^*}{\partial \beta_i} (\det(\mathbf{B}))^2 = (\det(\mathbf{B}) - A_i)\beta_i \left(\det(\mathbf{B}) - \beta_i \frac{\partial \det(\mathbf{B})}{\partial \beta_i} \right) > 0.$$

□

Theorem 4. *For all* $i, j \in \mathcal{K}$ *and* $\beta_i \neq 0$, $\frac{\partial p_i^*}{\partial \beta_j} < 0$.

Proof. We have that

$$
\begin{aligned}
\frac{\partial p_i^*}{\partial \beta_j}(\det(\mathbf{B}))^2 &= \frac{\partial(\det(\mathbf{B})-A_i)}{\partial \beta_j}\det(\mathbf{B}) - (\det(\mathbf{B})-A_i)\frac{\partial \det(\mathbf{B})}{\partial \beta_j} \\
&= \left((K-1)\beta_i \sum_{S\subseteq \mathcal{K}\backslash i,j}(-1)^{|S|+1}\prod_{l\in S}\beta_l\right)\left(\sum_{S\subseteq \mathcal{K}\backslash i}(-1)^{|S|+1}\left(|S|-1-\beta_i|S|\right)\prod_{l\in S}\beta_l\right) \qquad (3)\\
&- \left((K-1)\beta_i \sum_{S\subseteq \mathcal{K}\backslash i}(-1)^{|S|}\prod_{l\in S}\beta_l\right)\left(\sum_{S\subseteq \mathcal{K}\backslash i,j}(-1)^{|S|}\left(|S|-\beta_i(|S|+1)\right)\prod_{l\in S}\beta_l\right) \\
&= (K-1)\beta_i \left[\begin{array}{c} \left(\sum_{S\subseteq \mathcal{K}\backslash i,j}(-1)^{|S|+1}\prod_{l\in S}\beta_l\right)\left(\sum_{S\subseteq \mathcal{K}\backslash i}(-1)^{|S|+1}\left(|S|-1-\beta_i|S|\right)\prod_{l\in S}\beta_l\right) \\ -\left(\sum_{S\subseteq \mathcal{K}\backslash i}(-1)^{|S|}\prod_{l\in S}\beta_l\right)\left(\sum_{S\subseteq \mathcal{K}\backslash i,j}(-1)^{|S|}\left(|S|-\beta_i(|S|+1)\right)\prod_{l\in S}\beta_l\right)\end{array}\right]
\end{aligned}
$$

We next show that all terms containing β_j cancel out within the bracket. Fix $T, T' \subseteq \mathcal{K}\backslash i, j$ and consider the term $\beta_j \prod_{l\in T}\beta_l \prod_{l\in T'}\beta_l$. It appears in four different conditions:

i) if in $\frac{\partial(\det(\mathbf{B})-A_i)}{\partial\beta_j}$, $S = T$ and in $\det(\mathbf{B})$, $S = T' \cup j$:
the associated coefficient is $(-1)^{|T|+1} * (-1)^{|T'|+1+1}\left(|T'|+1-1\right) = (-1)^{|T|+|T'|+1}|T'|$
ii) if in $\frac{\partial(\det(\mathbf{B})-A_i)}{\partial\beta_j}$, $S = T'$ and in $\det(\mathbf{B})$, $S = T \cup j$:
the associated coefficient is $(-1)^{|T'|+1} * (-1)^{|T|+1+1}\left(|T|+1-1\right) = (-1)^{|T|+|T'|+1}|T|$
iii) if in $(\det(\mathbf{B})-A_i)$, $S = T\cup j$ and in $\frac{\partial\det(\mathbf{B})}{\partial\beta_j}$, $S = T'$:
the associated coefficient is $(-1)^{|T|+1} * (-1)^{|T'|}\left(|T'|\right) = (-1)^{|T|+|T'|+1}|T'|$
iv) if in $(\det(\mathbf{B})-A_i)$, $S = T'\cup j$ and in $\frac{\partial\det(\mathbf{B})}{\partial\beta_j}$, $S = T$:
the associated coefficient is $(-1)^{|T'|+1} * (-1)^{|T|}\left(|T|\right) = (-1)^{|T|+|T'|+1}|T|$
Given that the coefficients in iii), iv) are subtracted from those in i), ii), they cancel out.

Finally, fix $T, T' \subseteq \mathcal{K}\backslash i, j$ and consider the term $\beta_i\beta_j \prod_{l\in T}\beta_l \prod_{l\in T'}\beta_l$. It appears in four different conditions:

i) if in $\frac{\partial(\det(\mathbf{B})-A_i)}{\partial\beta_j}$, $S = T$ and in $\det(\mathbf{B})$, $S = T' \cup j$:
the coefficient associated is $(-1)^{|T|} * (-1)^{|T'|+1}\left(-(|T'|+1)\right) = (-1)^{|T|+|T'|}(|T'|+1)$
ii) if in $\frac{\partial(\det(\mathbf{B})-A_i)}{\partial\beta_j}$, $S = T'$ and in $\det(\mathbf{B})$, $S = T \cup j$:
the coefficient associated is $(-1)^{|T'|} * (-1)^{|T|+1}\left(-(|T|+1)\right) = (-1)^{|T|+|T'|}(|T|+1)$
iii) if in $(\det(\mathbf{B})-A_i)$, $S = T\cup j$ and in $\frac{\partial\det(\mathbf{B})}{\partial\beta_j}$, $S = T'$:
the coefficient associated is $(-1)^{|T|+1} * (-1)^{|T'|}\left(-|(T'|+1)\right) = (-1)^{|T|+|T'|}(|T'|+1)$
iv) if in $(\det(\mathbf{B})-A_i)$, $S = T'\cup j$ and in $\frac{\partial\det(\mathbf{B})}{\partial\beta_j}$, $S = T$:
the coefficient associated is $(-1)^{|T'|+1} * (-1)^{|T|}\left(-(|T|+1)\right) = (-1)^{|T|+|T'|}(|T|+1)$
Again, it cancels out.

Given this simplification, we can then rewrite equation (3) as follows:

$$\begin{aligned}
\frac{\partial p_i^*}{\partial \beta_j}\left|\mathbf{B}_N\right|^2 &= (K-1)\beta_i \left[\begin{array}{c} \left(\sum_{S\subseteq\mathcal{K}\setminus i,j}(-1)^{|S|+1}\prod_{l\in S}\beta_l\right)\left(\sum_{S\subseteq\mathcal{K}\setminus i}(-1)^{|S|+1}\left(|S|-1-\beta_i\left|S\right|\right)\prod_{l\in S}\beta_l\right) \\ -\left(\sum_{S\subseteq\mathcal{K}\setminus i}(-1)^{|S|}\prod_{l\in S}\beta_l\right)\left(\sum_{S\subseteq\mathcal{K}\setminus i,j}(-1)^{|S|}\left(|S|-\beta_i(|S|+1)\right)\prod_{l\in S}\beta_l\right) \end{array}\right] \\
&= -(K-1)\beta_i \left[\begin{array}{c} \left(\sum_{S\subseteq\mathcal{K}\setminus i,j}(-1)^{|S|}\prod_{l\in S}\beta_l\right)\left(\sum_{S\subseteq\mathcal{K}\setminus i,j}(-1)^{|S|+1}\left(|S|-1-\beta_i\left|S\right|\right)\prod_{l\in S}\beta_l\right) \\ -\left(\sum_{S\subseteq\mathcal{K}\setminus i,j}(-1)^{|S|}\prod_{l\in S}\beta_l\right)\left(\sum_{S\subseteq\mathcal{K}\setminus i,j}(-1)^{|S|+1}\left(|S|-\beta_i(|S|+1)\right)\prod_{l\in S}\beta_l\right) \end{array}\right] \\
&= -(K-1)\beta_i\left(\sum_{S\subseteq\mathcal{K}\setminus i,j}(-1)^{|S|}\prod_{l\in S}\beta_l\right)\left(\sum_{S\subseteq\mathcal{K}\setminus i,j}(-1)^{|S|}(1-\beta_i)\prod_{l\in S}\beta_l\right) \\
&= -(K-1)\beta_i(1-\beta_i)\left(\sum_{S\subseteq\mathcal{K}\setminus i,j}(-1)^{|S|}\prod_{l\in S}\beta_l\right)^2 < 0,
\end{aligned}$$

given that $\beta_i \in \left(0, \frac{1}{K-1}\right]$ and $K > 1$. □