Harvesting a Remote Renewable Resource

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Abstract

In standard models of spatial harvesting, the resource is distributed over the complete domain and the agent is able to control the harvesting activity everywhere all the time. In some cases though, it is more realistic to assume that the resource is located at a single point in space and that the agent is required to travel there in order to be able to harvest. In this case, the agent faces a combined travelling–and–harvesting problem. We scrutinize this type of a two-stage optimal control problem, and illuminate the interdependencies between the solution of travelling and that of the harvesting sub-problem. Since the model is parsimoniously parameterised, we are able to analytically characterise the optimal policy of the complete travelling–and–harvesting problem. In an appendix we show how bounds on either control, *i. e.* on acceleration and on the harvesting capacity, as well as a positive discount rate affect the solution of the travelling–and–harvesting problem.

Keywords: Optimal travelling–and–harvesting decision; spatial renewable resource; continuous time; optimal control; two-stage control problem

JEL classification: Q20, Q22, C61

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1. Introduction

In the management of renewable natural resources the spatial dimension has attracted substantial attention in the last years. The focus of this literature is on the movement of the resource, such as fish or game, and on the optimal allocation of the harvesting effort over the domain (distributed control). In this paper, we reverse this view: we consider an agent required to move within the space in order to be able to harvest an immobile resource. Since travelling is a pre-requisite of harvesting, there is a mutual interaction between the travelling and the harvesting policy—and it is this spatio-temporal interaction of both policies that this paper focuses on.

The management of renewable natural resources is a central issue in economics since Gordon (1954) and Smith (1968) have advanced this topic. In this respect, optimal control theory has proven to be a suitable technique to design optimal harvesting strategies. Notably, in their monographs Conrad and Clark (1987), Conrad (2010) and Clark (2010) nicely demonstrate how optimal control theory may constructively contribute to the management of fisheries. Subsequently, these textbook models have been extended and generalised in various respects. For example, Fan and Wang (1998) generalise the optimal harvesting policy of an autonomous harvesting problem with logistic growth (see, for example, Clark, 2010) to a non-autonomous case with periodic coefficients; Liski et al. (2001) accounting for costly changes of the harvesting rate, explore the effects of increasing returns to scale for a standard fishery management model;[1] and Ainseba et al. (2003), Feichtinger et al. (2003), Hritonenko and Yatsenko (2006), Tahvonen (2008, 2009a,b), Li and Yakubu (2012), Skonhoft et al. (2012), Quaas et al. (2013), Tahvonen et al. (2013) and Belyakov and Veliov (2014) investigate harvesting of age-structured populations.[2]

While that work takes into account the temporal and the bioeconomic dimension, the spatial dimension—though already present in the literature of theoretical biology and applied mathematics—has entered the focus of economists relatively late: only in 1999 Sanchirico and Wilen brought the spatial dimension to the attention of resource economists. In their seminal paper, Sanchirico and Wilen (1999)

[1]In this way, these authors demonstrate a link between stable limit cycle policies and increasing returns in harvesting; notably, they show that for moderate adjustment costs the harvest rate and thus the stock of fish may oscillate persistently.

[2]Notably, Ainseba et al. (2003), investigating the optimal harvesting problem for a non-linear age-dependent and spatially structured population dynamics model, prove the existence and uniqueness of a solution along with the existence of an optimal control, and provide necessary optimality conditions.

generalize the fundamental open-access models of Gordon (1954) and Smith (1968) in the spatial direction: they set up a bioeconomic model with a finite number of resource patches with migration of the biomass and reallocation of effort between these patches. In this way the authors integrate within- and between-patch biological and economic forces, and demonstrate how these effects determine the process of bioeconomic convergence over space and time.[3]

Following Sanchirico and Wilen (1999), the early models in spatial resource economics feature discrete patches, where the resource the stock evolves according to an ordinary differential equation (ODE) at each location; migration of the biomass is then modelled as entry and exit of the biomass from one location to the other. The contemporary literature however, models the migration and the spread of the biomass as diffusion described by partial differential equations.[4] Notable contributions are Cañada et al. (1998), Montero (2000, 2001), Neubert (2003), Bai and Wang (2005), Brock and Xepapadeas (2008, 2010), Ding and Lenhart (2009), Joshi et al. (2009), Bressan et al. (2013), Uecker and Upmann (2016) and others.

In both strands of the literature it is the biomass which is mobile while the agent harvesting the resource is immobile: the agent is waiting for the resource approaching, catching it when passing by. In many instances this is a reasonable approach suitably describing the situation (*e. g.* coastal fishery or shooting game), but in other cases it is not. For example, in fruit harvesting, forestry, extensive agriculture *etc.* it is the agent who is moving in order to access the resource that is located at some fixed known patch. In this paper, we build on that observation and analyse the optimal behaviour of an agent who is required to travel in order to be able to harvest a remote resource. Thus, when compared with the standard approach in spatial resource economics with a mobile resource, we complement that literature by reversing the abilities of movement. This reversal enhances the realism in modelling natural resource extraction when the resource is rather immobile and is located at some distant or hardly accessible place, or when the spatial domain is relatively large compared with the region which can be harvested at a single instant of time.

In order to move from their initial location to the location of the resource and then to harvest, the agent first has to control the navigation process and then, upon arrival at the resource, the harvesting process. Consequently, any admissible

[3]In a subsequent paper Sanchirico and Wilen (2005) utilize the model of their 1999 paper to characterise the spatially differentiated landings and effort taxes suitable to implement a first-best allocation.

[4]A presentation of the basic population models with diffusion can be found, for example, in Aniţa (2000, sec. 1.2), Okubo and Levin (2001), Murray (2003) and the references therein.

policy consists of a sequence of a travelling and a harvesting interval—and we are interested in the interdependency of the travelling and the harvesting decision. By considering this sequence of time periods required for travelling and harvesting, we complement the contemporary literature on spatial resources economics.

Few papers consider a travelling–and–harvesting problem of the agent in a spatial domain. Notable examples are Robinson et al. (2008), Behringer and Upmann (2014) and Belyakov et al. (2015) who consider an immobile resource located at known patches. Behringer and Upmann and Belyakov et al. analyse an immobile resource that is continously distributed on the periphery of a circle and an agent who leaves for a round trip, returning home after each turn. In both models, the agent is able to do *en passant* harvesting, so that the agent needs not reduce speed or stop (at each location) in order to extract the resource; rather, the agent is able to extract the resource in passing by. As a consequence, the harvesting activity does not cost any time, over and above the time of travelling, but can be done during travelling. In this way, the travelling and the harvesting activity go in parallel and may even be identified with each other. This is opposite to our approach here, where travelling and harvesting are mutually exclusive, rival activities (with different cost functions): the more time is spent on travelling, the less time is left for harvesting, and *vice versa*.

Robinson et al. (2008) provide a timber gathering model which has parallels with our paper. Their model also treats discrete resource patches and assumes that travelling and harvesting are exclusive as we do in this paper. While we focus on the harvesting and the travelling process themselves, each governed by an independent control for which we derive analytical solutions, in Robinson et al. (2008) the decision about travelling speed and the amount harvested are connected via a gathering-specific cost function; those authors formalize the idea that the cost of (additional) travelling time is increasing in the amount already harvested, as this yield has to be carried back to a nearby market.[5]

Since in our approach travelling and harvesting are two distinct activities which take place at different locations at different times, we are confronted with two interdependent optimal control problems: the problem of travelling, where speed has to be chosen to travel to the location of the resource; and the harvesting problem, where the harvesting rate has to be determined to maximise the yield. In order to solve this combined profit-maximising problem, we draw upon the literature of two-stage optimal control problems with finite time horizon: notably,

[5]Such a market where the resource is eventually traded in a continuous setting is the focus of Aniţa et al. (2017).

Amit (1986), Tomiyama (1985), Tomiyama and Rossana (1989) provide optimality conditions for two-phase, finite time dynamic optimization problems similar to the one considered here.[6]

We solve this two-stage optimal control problem and derive the optimal travelling–and–harvesting policy, including the optimal point in time at which the agent arrives at the location of the resource and begins harvesting. In particular, we demonstrate the interdependency between the travelling and the harvesting problem, a feature which has, to our knowledge, been left unnoticed and unexplored in the literature. To scrutinize the robustness of our finding, we consider two different specifications of the growth process of the resource: exponential growth and logistic growth. For both types of processes we derive the optimal harvesting policies, and show that both feature similar characteristics. Finally, we briefly investigate the sensitivity of our results with respect to the rate at which future revenues and costs are discounted and with respect to the presence of bounds on the control of movement.

The rest of the paper is structured as follows: In Section 2 we set up the model. In Section 3 we decompose the travelling–and–harvesting problem into the two sub-problems. We begin our analysis with the harvesting problem in Section 4; and then consider the travelling–and–harvesting problem for a fixed travelling period in Section 5, before we analyse the full problem with a variable travelling period in Section 6. We conclude in Section 7. The robustness of our results are explored in Appendix A.

2. The Model

We consider a renewable natural resource located at some fixed location. The agent can harvest the resource at their current location only, and is thus required to travel in order to get access to, and to be able to extract the resource. Consequently, the agent may begin to harvest upon arival at the location of the resource, at the earliest. Since the process of harvesting takes time, the stock diminishes gradually while harvesting takes place. The agent's problem is thus a combined travelling–and–harvesting problem where the speed of travelling, and hence the arrival time, and the harvesting rate have to be determined jointly in order to maximise the total profit composed of the revenue from harvesting net of harvesting and travelling cost.

[6]An extension to infinite horizon can be found in Makris (2001); and applications of this theory to two-stage optimal control problems, in Grass et al. (2012), Bar-Ilan and Strange (1998), Tahvonen and Withagen (1996) and Boucekkine et al. (2004), for example.

We consider a finite time horizon T with a planning period $\mathscr{T} \equiv [0, T]$. During this planning period, the economic agent has the exclusive right to harvest the renewable natural resource, which is located at some fixed and known position $x_1 > 0$. At time $t \in \mathscr{T}$ the location of the economic agent is $x(t) \in \mathscr{X} \equiv [0, \bar{x}]$, with $x_1 \leq \bar{x}$; with the initial location given by $x(0) = 0$. Since the resource is remotely located, at a distance of x_1 units of length from the agent, the agent is unable to begin with harvesting until they arrive at location $x(t) = x_1$. Harvesting thus requires the agent to travel from 0 to x_1, to stop there, and to begin with harvesting.

In order to move from one location to the next, the agent has to adjust the velocity of travelling $v(t)$, which we assume to be non-negative, *i. e.* $v(t) \in \mathscr{V} = \mathbb{R}_+$.[7] Since speed cannot be chosen directly, but is physically controlled by means of acceleration $a(t) \in \mathscr{A}$ of the vehicle of movement or the harvesting machine we have[8]

$$\dot{x}(t) = v(t) \quad \text{and} \quad \dot{v}(t) = a(t) \quad \forall t \in \mathscr{T}. \tag{1}$$

There may be lower and upper bounds on acceleration; in Appendix A, we shall assume that acceleration is bounded so that $a \in \mathscr{A} \equiv [\underline{a}, \bar{a}]$ with $\underline{a} < 0$ and $\bar{a} > 0$.[9]

Since harvesting, as well as travelling, takes time and the time horizon is finite, more time is left for harvesting the earlier the agent arrives at location x_1. More precisely, let $t_1 \equiv \min_t\{t | x(t) = x_1\}$ denote the arrival time of the agent at the location of the resource x_1, that is $x(t) < x_1$ for all $t < t_1 \leq T$ and $x(t_1) = x_1$; if the agent does not arrive at x_1 by time T, such that $x(T) < x_1$, then we set $t_1 = +\infty$. Thus, we have $t_1 \in \mathscr{T}_{+\infty} \equiv \mathscr{T} \cup \{+\infty\}$. Consequently, $\Lambda \equiv [0, t_1 \wedge T]$ denotes the agent's travelling period;[10] and $\Delta \equiv (t_1 \wedge T, T]$, the resulting harvesting period. The total time available is then either spent on travelling or on harvesting, so that $\Lambda \cup \Delta = \mathscr{T}$ represents the *travelling–and–harvesting period*; this is illustrated in Figure 1.

The stock of the resource (*i. e.* the biomass) at time $t \in \mathscr{T}$ is denoted by $s(t) \geq 0$. We assume that the renewable resource is growing at rate g, and allow for the growth rate of the stock to depend on the size of the stock: $g(s)$ with

[7]The assumption of non-negative speed rules out that the agent moves backwards. Since moving backwards is economically unreasonable, this assumption can be made without restrictions.

[8]Taking into account acceleration avoids an unrealistic speed profile where the agent may instantaneously switch speed in a discontinuous way.

[9]The minimum acceleration $\underline{a}$ is necessarily negative to allow for a slowdown of speed, as the agent would otherwise be unable to stop—and start harvesting.

[10]We assume that the travelling period is convex. That is, once the agent has reached location x_1, they will never start travelling again, and thus completes the planning period at x_1.

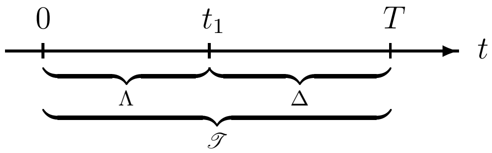

FIGURE 1. Travelling–and–harvesting period

$g(0) = 0$. Furthermore, the stock is reduced as a result of the harvesting activity. The harvest depends on the abundance of the resource, *i. e.* on the stock s, and on the harvesting effort h. Suppose that effort is less productive the lower the stock, and that a given stock yields less harvest the lower the effort. Accordingly, we assume that harvest at time t amounts to $H(t) = h(t)s(t)$ provided that the agent's location is x_1, and $H(t) = 0$ otherwise. Thus, the resulting growth of the stock is governed by the differential equation

$$\dot{s}(t) = g(s(t)) - h(t)s(t)\mathbf{1}_{\{x(t)=x_1\}}(t), \quad \forall t \in \mathscr{T}, \tag{2}$$

where the indicator function $\mathbf{1}_{\{x(t)=x_1\}}$ accounts for the fact that harvesting can only be effective if the agent's location at time t equals x_1, *i. e.*, if $x(t) = x_1$. In other words, upon arrival at location x_1, the agent starts the path of the harvesting activity $\{h(t)\}_{t\in\Delta}$.

The process of harvesting gradually diminishes the stock, and the agent may decide to continue harvesting until the stock is depleted: with $s(t) = 0$ it immediately follows that $H(t) = 0$ for any harvesting activity $h(t) \geq 0$. Also, once the stock is depleted, we have $\dot{s}(t) = 0$ due to our assumption $g(0) = 0$. Hence, $s = 0$ represents an absorbing barrier or an equilibrium of the stock dynamics. (Subsequently, we will consider the cases of exponential and logistic growth, both of which satisfy these assumptions.) Owing to the immediate, negative effect of harvesting on growth, intensive harvesting leaves the stock with less beneficial conditions for future growth, and thus impairs the possibilities for future harvesting.

Travelling and harvesting are both costly. We assume that harvesting cost $C(H)$ is increasing and (weakly) convex, *i. e.* $C' > 0$ and $C'' \geq 0$ for all $H \in \mathbb{R}_+$, with $C(0) = 0$. Also, travelling is associated with some cost, which generically depends on both speed and acceleration: $K : \mathscr{V} \times \mathscr{A} \to \mathbb{R} : (v, a) \mapsto K(v, a)$. Naturally, pausing is costless, $K(0, 0) = 0$; and we assume that travelling cost increases with both speed and acceleration, and that acceleration is more costly the higher the speed, that is, the partial derivatives of K satisfy $K_v \geq 0, K_a \geq 0$ and $K_{va} \geq 0$.

Let $\rho \geq 0$ denote the discount rate of the agent, and normalize the price of one unit of the harvested resource to unity. The problem of the agent is then to maximize the discounted profit flow consisting of instantaneous revenue net of harvesting cost and net of travelling cost for the planning period $\mathscr{T}$. Presupposing that the agent reasonably chooses $h(t) = 0, \forall t \in \Lambda$,[11] we obtain the travelling cost

$$J_1(a, t_1) \equiv \int_0^{t_1} e^{-\rho t} K(v(t), a(t))\, \mathrm{d}t \tag{3}$$

and the profit from harvesting

$$J_2(h, t_1) \equiv \int_{t_1}^{T} e^{-\rho t} \left(h(t)s(t) - C(h(t)s(t)) \right)\, \mathrm{d}t \tag{4}$$

where the arrival time t_1 depends on the acceleration path $\{a\}_{t \in \Lambda}$. Putting pieces together, the agent's optimisation problem then reads as

$$\max_{\{a,h\}} \quad J(a, h, t_1) \equiv -J_1(a, t_1) + J_2(h, t_1) \tag{5}$$

subject to the dynamics of movement (1), the stock dynamics of the resource (2), and their associated constraints $v(t) \in \mathscr{V}(t)$, $a(t) \in \mathscr{A}(t)$, $h(t) \in \mathscr{H}(t)\ \forall t \in \mathscr{T}$, as well as to the initial conditions $s(0) = s_0$, $x(0) = 0$ and $v(0) = 0$, the "arrival conditions" $t_1 \in \mathscr{T}_{+\infty}$ free, $x(t_1) = x_1$ and $v(t_1) = 0$ if $t_1 \in \mathscr{T}$, and the terminal condition $s(T) \geq 0$ free, $x(T) \in \mathscr{X}$ free. Note that the constraints $x(t) = x_1, \forall t \in \Delta$ and $v(t) = 0, \forall t \in \Delta$ are already implied by (2) and thus need not be added.[12]

3. Decomposition of the problem

In order to solve problem (5), we decompose the intertemporal optimal travelling–and–harvesting problem into a travelling and a harvesting sub-problem. In order to render the problem meaningful, we subsequently assume that the costs of travelling are not too high, so that an arrival before time T is desirable. In addition, because maximum acceleration is finite, the arrival time must be strictly positive. Therefore, the corner solutions $t_1 = 0$ and $t_1 = T$ (as well as $t_1 = +\infty$) can be ruled out, so that neither the travelling period nor the harvesting period vanishes, *i. e.* $\Lambda, \Delta \neq \emptyset$. For those reasons, we subsequently presume $t_1 \in (0, T)$.

[11] In principle, we allow the agent to choose $h(t) > 0$ for times $t < t_1$, but since this harvesting activity is sure to yield no return at any time $t \in \Lambda$, the choice of $h(t) > 0$ is a futile action in this case.

[12] Also, $v(T) = 0$ is automatically fulfilled for any optimal policy.

In the *travelling problem* we choose an acceleration path $\{a(t)\}_{t\in\Lambda}$ and thus the arrival time t_1 so as to move from location 0 to location x_1 at minimal cost:

$$
\begin{aligned}
&\min_{\{a,t_1\}} J_1(a,t_1) \equiv \int_0^{t_1} e^{-\rho t} K(v(t),a(t))\,\mathrm{d}t && && (6)\\
&\text{s.t.} \quad \dot{x}(t) = v(t), && && \forall t \in [0,t_1]\\
&\qquad \dot{v}(t) = a(t), && && \forall t \in [0,t_1]\\
&\qquad a(t) \in \mathscr{A}(t), && && \forall t \in [0,t_1]\\
&\qquad v(t) \in \mathscr{V}(t), && && \forall t \in [0,t_1]\\
&\qquad \dot{s}(t) = g(s(t)), && && \forall t \in [0,t_1]\\
&\qquad x(0) = x_0, && x(t_1) = x_1, &&\\
&\qquad v(0) = 0, && v(t_1) = 0, &&
\end{aligned}
$$

Since the travelling time t_1 can be chosen, we face a free-terminal-time problem. Then, t_1 represents the starting time of the harvesting period, and in the resulting *harvesting problem* we choose a path of the harvesting activity (effort) $\{h(t)\}_{t\in\Delta}$ to maximise profit from this activity (4):

$$
\begin{aligned}
&\max_{\{h\}} J_2(h,t_1) \equiv \int_{t_1}^{T} e^{-\rho t}\left[h(t)s(t) - C(h(t)s(t))\right]\,\mathrm{d}t && (7)\\
&\text{s.t.} \quad \dot{s}(t) = g(s(t)) - h(t)s(t) && \forall t \in [t_1,T]\\
&\qquad h(t) \in \mathscr{H}(t), && \forall t \in [t_1,T]\\
&\qquad s(t_1) = s_1, \quad s(T) \geq 0, \text{ free.} &&
\end{aligned}
$$

During the travelling time the resource remains unimpaired and thus grows (at least) until the agent arrives at the location of the resource, x_1. Consequently, the stock of the resource at the time of arrival, $s(t_1)$, represents the solution of the growth process $\dot{s}(t) = g(s(t))$ with $s(0) = s_0$. In this way, the travelling decision determines $s(t_1)$ and thus the initial value of the stock of the harvesting problem. The fact that the travelling time of the agent also represents the growth time of the resource is the crucial link between the travelling problem (6) and the harvesting problem (7). As a consequence, the agent has to take into account that a longer travelling time will reduce the time left for harvesting, and thus *ceteris paribus* the resulting yield; while, in contrast, a lower speed of travelling makes travelling less expensive and gives the resource more time to grow thus providing the opportunity for a more abundant harvest at later times. The *travelling–and–harvesting problem* (5) takes into account these interdependencies between sub-problems (6) and (7).

To solve problem (5), we derive necessary conditions for an optimal control pair (a^*, h^*, t_1^*), by means of decomposing the original problem into two standard problems. We first consider the harvesting problem of the second stage (7), and then the travelling problem of the first stage (6), acknowledging the dependence of the solution of the second stage on the decision of the first stage. The interdependency comes about because the optimal control h^* of the harvesting problem depends on the choice of the starting value $s_1 = s(t_1)$ and the starting time t_1 determined by the solution of the travelling problem. Formally we proceed as follows: Assuming the existence of the optimal switching time t_1 in the interior of the time interval $\mathscr{T}$, we solve the second stage problem and calculate the maximised objective function J_2^* as a function of the initial state s_1 and the switching time t_1. Then, we derive the optimal control a^* and the optimal switching time t_1 by solving the travelling problem of the first stage.[13]

Second stage. Given the control time interval $[t_1, T]$ and the initial condition $s(t_1) = s_1$, we solve problem (7) for an admissible optimal control h^*. This problem is of a standard form and can be solved using the well-known Pontryagin maximum principle (see, for example, Kamien and Schwartz, 1991.) Using the solution of the second-stage problem, h^*, λ_2^* and s^*, which depends on the starting values s_1 and t_1, we calculate $J_2^*(s_1, t_1) \equiv J_2(h^*(s_1, t_1), t_1)$. Then, with the help of J_2^*, the original problem (5) reduces to the *first–stage problem*:

First stage. Given the constraints in (6), we look for an admissible optimal control a^* defined on $[0, t_1^*]$ and an optimal arrival time $t_1^* \in (0, T)$ such that

$$\max_{\{a, t_1\}} V_1(a, t_1) \equiv -J_1(a, t_1) + J_2^*(s(t_1), t_1). \tag{8}$$

Since by assumption $t_1^* \in (0, T)$, the constraint $t_1 \in (0, T)$ is irrelevant, and this problem reduces to a standard problem with 'scrap' value J_2^*, free terminal time t_1 and end point $s(t_1)$. (See, for example, Léonard and Long, 1992, sec. 7.2 and 7.6.)

The optimality conditions for this type of a two-phase dynamic optimization problem are available from the literature. Details can be found in Tomiyama (1985) and Amit (1986) who provide necessary conditions for a two-stage, finite-horizon switching problem with endogenous switching time; while Makris (2001)

[13] Tomiyama and Rossana (1989) and Grass et al. (2008, sec. 8.1.1) generalise the results of Tomiyama (1985) and Amit (1986) for a finite and an infinite time horizon, respectively, when the switching point appears as an argument of the integrands in each integral of the objective function.

provides corresponding results for a two-stage switching problem with an infinite time horizon. We here apply the results of Tomiyama (1985) and Amit (1986).

4. Second Stage: Harvesting

We now solve the travelling–and–harvesting problem in the suggested way, *i. e.* we solve the harvesting problem in this section, and then solve the problem of the second stage in Section 6. We consider two standard specifications of the growth process of the resource: exponential growth in sub-section 4.1, and logistic growth in sub-section 4.3. For both processes we derive the value function of the optimal harvesting policy. (Similar models can be found in Conrad and Clark, 1987; Hocking, 1991; Clark, 2010.) To be specific, we subsequently speak of fish and catch, though the analysis is fully applicable to other remote natural renewable resources.

4.1. Exponential growth. Suppose that the stock of a given species of fish, when left unimpaired, increases at a constant rate: $g(s(t)) = s(t)$ for all $t \in \Delta \equiv [t_1, T]$. Since the stock is reduced by the catch $H(t) \equiv s(t)h(t)$, the stock evolves according to the differential equation

$$\dot{s}(t) = s(t) - h(t)s(t), \quad s(t_1) = s_1, \quad \forall t \in \Delta, \forall h(t) \in \mathscr{H}. \tag{9}$$

We follow the familiar Schaefer model (see Schaefer, 1954), and specify the revenue from fishing as a bi-linear function of effort and the stock $H(t) = qs(t)e(t)$, where q is the catchability coefficient, defined as the fraction of the population fished by means of a unit of effort; for convenience we set $q = 1$. Also, concordantly with the literature, we presuppose a constant price of the resource so that revenue amounts to $pH(t)$. To complete our definition of the profit function, we follow the specification of the effort cost function chosen by, for example, Puchkova et al. (2014) and Moberg et al. (2015) and assume that harvesting costs are linear in total catch, $C(H(t)) = cH(t) = c\,h(t)s(t)$, with $0 \leq c < p$.[14] Then, instantaneous

[14] The Schaefer model is commonly used in the literature, and many authors add either linear or quadratic effort cost. For example, Clark (2010, Sec. 1.4), Puchkova et al. (2014) and Moberg et al. (2015) assume linear cost yielding an instantaneous profit equal to $pqs(t)e(t) - ce(t)$; while He et al. (1994), Leung (1995), Cañada et al. (2001), Montero (2001), Fister and Lenhart (2004, 2006) and Chang and Wei (2012) presume a quadratic effort cost function, and Ding and Lenhart (2009) presume a linear-quadratic effort cost function. One exception to the prevalence of linear and quadratic cost functions is Liski et al. (2001) who suppose a concave-convex harvest cost. An alternative specification of the objective function is to disregard effort cost altogether and to maximise the sustainable yield; this approach is followed by, for example, Fan and Wang (1998), Neubert (2003), Bai and Wang (2005) and Kelly et al. (2016).—All of these authors assume a fixed price of the (harvested) resource.

profit amounts to $(p-c)h(t)s(t)$. Finally, we normalize the per unit profit $p-c$ to unity, so that the objective function becomes

$$\max_{\{h\}} J_2(h, t_1) = \int_{t_1}^{T} h(t)s(t)\,\mathrm{d}t \quad \text{s.t. (9)}. \tag{10}$$

Following a substantial part of the literature, we abstract from discounting for the moment, and set $\rho = 0$. (For example, the majority of the references provided in fn. 14 abstracts from discounting.) This allows us to simplify the analysis, and we show in Appendix A, how our results are affected by the presence of a positive discount rate.

The Hamiltonian of this problem is given by

$$\mathcal{H} = h(t)s(t) + \pi(t)s(t)\left(1 - h(t)\right), \tag{11}$$

and the maximum principle yields

$$0 \;=\; (1-\pi(t))s(t), \tag{12}$$

$$\dot{\pi}(t) \;=\; h(t)\pi(t) - h(t) - \pi(t), \tag{13}$$

along with eq. (9). Apparently, the optimal strategy depends on whether π is less or greater than one. The maximum of $\mathcal{H}$ is thus achieved by

$$h(t) = \begin{cases} 0 & \text{if } \pi(t) > 1 \\ \bar{h} & \text{if } \pi(t) < 1. \end{cases} \tag{14}$$

Since $s(T)$ is free, the transversality condition requires $\pi(T) = 0$. This, together with $h(t) = \bar{h}$ for $\pi(t) < 1$, implies that we cannot end the period Δ with $h = 0$, *i. e.* we must have $h(T) = \bar{h}$. Moreover, the solution of eq. (13) must satisfy

$$\pi(t) = \begin{cases} A_0 e^{-t} & \text{if } h(t) = 0 \\ \dfrac{\bar{h}}{\bar{h}-1} + A_1 e^{t(\bar{h}-1)} & \text{if } h(t) = \bar{h}. \end{cases} \tag{15}$$

Neither solution achieves the critical value $\pi = 1$ more than once. Consequently, there is a unique switching point τ,[15] implying that we either have (i) $h(t) = \bar{h}$ for all $t \in \Delta$, or (ii) $h(t) = 0$ for all $t_1 \le t < \tau$ and $h(t) = \bar{h}$ for all $\tau \le t \le T$. Then, along any path with $h = \bar{h}$, the costate variable must satisfy

$$\pi(t) = \frac{\bar{h}}{\bar{h}-1}\left(1 - e^{(1-\bar{h})(T-t)}\right) \tag{16}$$

[15] Alternatively, this observation follows from eq. (13), which implies that evaluated at a switching point τ we have $\dot{\pi}(\tau) = -1$ since $\pi(\tau) = 1$ by definition.

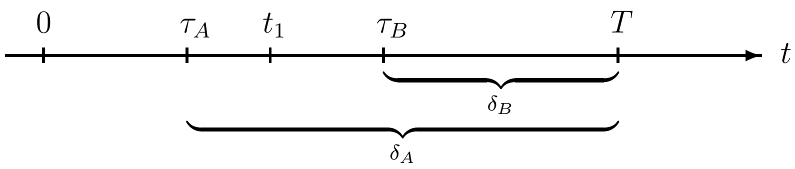

FIGURE 2. Cases A and B

where we determined $A_1 = \bar{h}e^{T(1-\bar{h})}/(1-\bar{h})$ so as to satisfy the transversality condition $\pi(T) = 0$. Now, the switching time τ has to be chosen according to the condition $\pi(\tau) = 1$. Hence, we obtain from eq. (16)

$$\tau = T - \delta, \quad \text{with} \quad \delta \equiv \frac{\log\left(\bar{h}\right)}{\bar{h}-1}. \tag{17}$$

Since δ is a positive, decreasing and convex function for all values of $\bar{h} \neq 1$, we define $\delta = 1$ for $\bar{h} = 1$ so as to make δ a continuous function of $\bar{h}$.[16] Consequently, the larger $\bar{h}$, the longer the agent can wait and let the resource grow unimpaired, allowing for more intensive harvesting later. Depending on the sign of $\tau - t_1$, either of two cases may occur.

4.1.1. *Case A:* $T < \delta + t_1$. In this case the maximal harvesting intensity $\bar{h}$ is relatively low requiring a rather long period of extraction: $T - t_1 < \delta \Leftrightarrow \tau < t_1$. This implies that there is no switch in policy and for all $t \in \Delta$, and thus we have:

Lemma 1. *Let* $T < \delta + t_1$ *and* $\bar{h} \neq 1$. *Then the optimal fishing policy is given by*

$$\begin{aligned} h(t) &= \bar{h}, & (18)\\ s(t) &= s_1 e^{\left(1-\bar{h}\right)(t-t_1)}, & (19)\\ \pi(t) &= \frac{\bar{h}}{\bar{h}-1}\left(1 - e^{\left(1-\bar{h}\right)(T-t)}\right), & (20)\end{aligned}$$

for all $t \in \Delta$. *The resulting maximised profit amounts to*

$$J_{2A}^*(s_1, t_1) \equiv s_1 \frac{\bar{h}}{\bar{h}-1}\left(1 - e^{\left(1-\bar{h}\right)(T-t_1)}\right). \tag{21}$$

4.1.2. *Case B:* $T > \delta + t_1$. In this case the maximal harvesting intensity $\bar{h}$ is relatively high so that the agent may afford not to begin with harvesting immediately at time t_1 but at some point in time: $T - t_1 > \delta \Leftrightarrow \tau > t_1$. Here the agent begins with $h = 0$ and then, at time τ, switches to $h = \bar{h}$. During the period $[t_1, \tau)$ the stock is left unimpaired and is thus given by $s(t) = s_1 e^{t-t_1}$, so that at time τ the

[16]To see that $\delta = 1$ for $\bar{h} = 1$, apply l'Hôpital's rule.

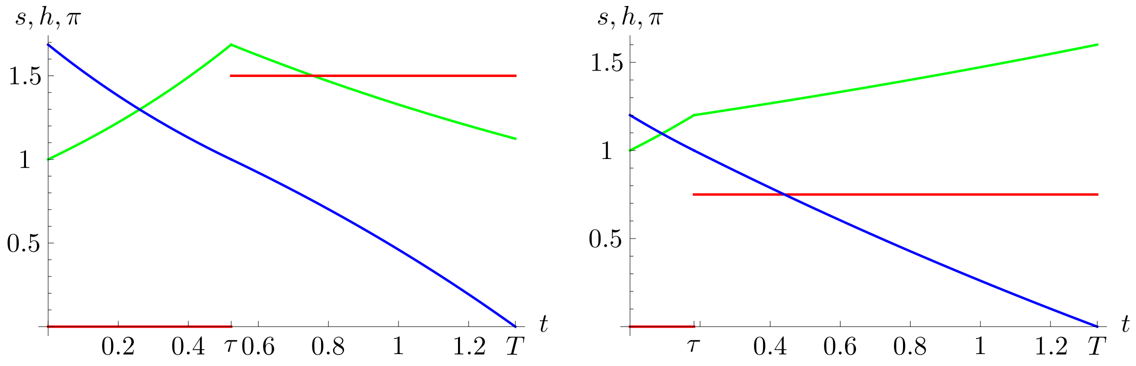

FIGURE 3. Optimal fishing in Case B, $T > \delta + t_1$, with $t_1 = 0$: $\bar{h} > 1$ (left) and $\bar{h} < 1$ (right).

stock amounts to $s(\tau) = s_1 e^{\tau - t_1}$, which is the starting value for the harvesting period $[\tau, T]$, so that for times $t \in [\tau, T]$ the stock equals

$$s(t) = A_2 e^{(1-\bar{h})t} = s(\tau) e^{\left(1-\bar{h}\right)(t-\tau)} = s_1 e^{\bar{h}(\tau - t) + t - t_1}.$$

Hence, for all $\bar{h} \neq 1$ the optimal policy is thus given by:

Lemma 2. *Let $T > \delta + t_1$ and $\bar{h} \neq 1$. Then the optimal fishing policy is given by*

$$h(t) = \begin{cases} 0 & \textit{for } t_1 \leq t < \tau \\ \bar{h} & \textit{for } \tau \leq t \leq T \end{cases}$$

$$s(t) = \begin{cases} s_1 e^{t - t_1} & \textit{for } t_1 \leq t < \tau \\ s_1 e^{\bar{h}(\tau - t) + t - t_1} & \textit{for } \tau \leq t \leq T \end{cases}$$

$$\pi(t) = \begin{cases} e^{\tau - t} & \textit{for } t_1 \leq t < \tau \\ \frac{\bar{h}}{\bar{h}-1}\left(1 - e^{\left(1-\bar{h}\right)(T-t)}\right) & \textit{for } \tau \leq t \leq T, \end{cases}$$

for all $t \in \Delta$, and the maximised profit amounts to

$$J_{2B}^*(s_1, t_1) \equiv \frac{\bar{h}}{\bar{h}-1} s_1 e^{\tau - t_1} \left(1 - e^{(1-\bar{h})(T-\tau)}\right) = s_1 \, \bar{h}^{1/(1-\bar{h})} \, e^{T - t_1}. \tag{22}$$

4.1.3. *The case of $\bar{h} = 1$.* Finally, the optimal policy for the case $\bar{h} = 1$ is obtained by taking the limits of Case A and Case B:

Remark 1. If $\bar{h} = 1$, the optimal profit amounts to

$$J_2^*|_{\bar{h}=1}(s_1, t_1) = \begin{cases} s_1(T - t_1) & \text{if } T \leq \delta + t_1 \\ s_1 e^{T - t_1 - 1} & \text{if } T > \delta + t_1. \end{cases} \tag{23}$$

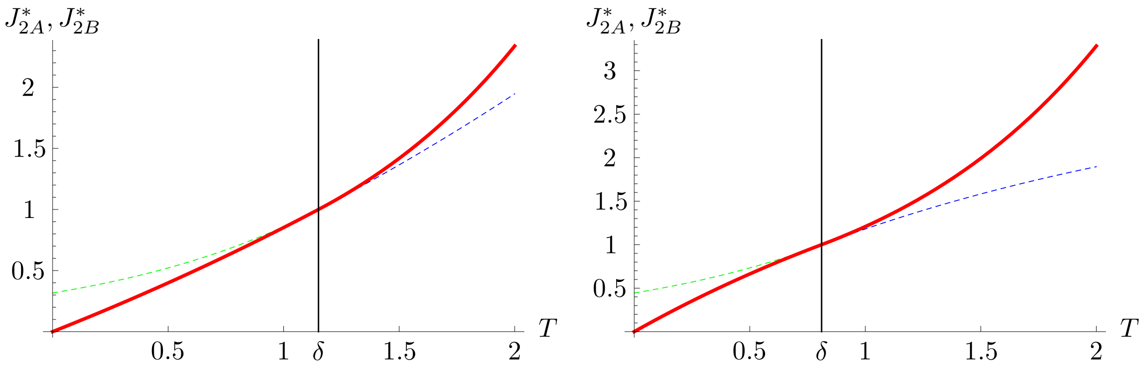

FIGURE 4. Maximised profit function for $\bar{h} = 3/4 < 1$, *i. e.* $\delta = 4\log\left(\frac{4}{3}\right) = 1.15073$ (left); and for $\bar{h} = 3/2 > 1$, *i. e.* $\delta = 2\log\left(\frac{3}{2}\right) = 0.81093$ (right).

4.2. Discussion. In the optimal solution, the length of the fishing period equals $\delta = T - \tau = \log(\bar{h})/(\bar{h} - 1)$. If there is plenty of time in the sense that $T > \delta + t_1$, there will be no fishing during the initial period of length $T - t_1 - \delta$, while fishing will take place at the maximum rate $\bar{h}$ during the final period. If, however, there is not enough time available, that is if $T \leq \delta + t_1$, the agent fishes all the time at the maximum rate $\bar{h}$. Whether the stock increases or decreases during fishing, depends on whether the harvesting capacity $\bar{h}$ exceeds or falls short of the growth rate of the stock, which is assumed to be equal to 1 here. The situation when $\bar{h}$ is smaller than 1 is depicted in the left diagram of Figure 3; and the situation with $\bar{h} > 1$, in the right diagram (both for $t_1 = 0$).

It is important to note that the optimal length of the fishing period, δ, depends on $\bar{h}$ but is independent of T. However, the maximised profit in Case A and B, given by eq. (21) and (22) respectively, depends on T. While J^*_{2B} is increasing and convex in T, J^*_{2A} is convex only if $\bar{h} < 1$, and is concave if $\bar{h} > 1$. Moreover, for any given values of t_1 and s_1 we have $J^*_{2B} \geq J^*_{2A}$. This is depicted in Figure 4 for the case $t_1 = 0$. Therein, the vertical line represents the critical time $T = \delta + t_1$ for a given value of $\bar{h}$, and the red curve depicts the profit function for varying values of T. If time is scarce in the sense that $T - t_1 < \delta$, Case A applies and the blue curve represents the resulting maximised profit (covered by the red curve for values $T < \delta$). If there is plenty of time, in the sense that $T > \delta + t_1$, Case B applies and the green curve represents the resulting maximised profit (similarly covered by the red curve for values $T > \delta + t_1$). Note, however, that in Case A, the Case B profit function is not attainable, so that the dashed green curve is merely hypothetical and cannot be reached for values of T lower than δ.

4.3. Logistic growth. In this section we modify the growth process of the resource and now assume that the stock obeys a logistic growth process:

$$f(s(t)) = 2s(t)\left(1 - \frac{s(t)}{2}\right). \tag{24}$$

With this specification, the net-growth of the stock, *i. e.* after deduction of the harvest, is governed by the differential equation

$$\dot{s}(t) = f(s(t)) - h(t)s(t) = s(t)\left(2 - s(t) - h(t)\right) \tag{25}$$

If left unimpaired, the fish stock equilibrates at the level $s^* = 2$. Let us assume that the initial stock equals that level, *i. e.* $s(t_1) = 2$.—The remaining model is adopted from Section 4.1.

The Hamiltonian of the problem is given by

$$\mathcal{H} = h(t)s(t) + \pi(t)s(t)\left(2 - s(t) - h(t)\right),$$

and the maximum principle yields

$$\begin{aligned} 0 &= (1 - \pi(t))s(t), & (26)\\ \dot{\pi}(t) &= -h(t) - \pi(t)\left(2 - 2s(t) - h(t)\right), & (27)\end{aligned}$$

along with eq. (25); and since $s(T)$ is free, the transversality condition requires $\pi(T) = 0$.

Lemma 3. $\pi(t_1) < 1$.

Proof. Assume, on the contrary, that $\pi(t_1) > 1$. Since $s(t_1) = 2$, it follows from eq. (27) that $\dot{\pi}(t_1) = -h(t_1) + \pi(t_1)\left(2 + h(t_1)\right) > 0$. Since $h(t) = 0$ as long as $\pi(t) > 1$, the stock remains at its starting value $s(t_1) = 2$. Given this, there is no turning point in the evolution of π and thus π continues to grow, *i. e.* we have $\dot{\pi}(t) > 0$ for all t. Yet, this contradicts transversality condition $\pi(T) = 0$, and thus proves our claim $\pi(t_1) < 1$, and thus $h(t_1) = \bar{h}$. □

It follows from Lemma 3 that the optimal policy rule coincides with the rule obtained for exponential growth of the resource (14):

Lemma 4. *The maximum of the Hamiltonian $\mathcal{H}$ is achieved by*

$$h(t) = \begin{cases} 0 & \text{if } \pi(t) > 1 \\ \bar{h} & \text{if } \pi(t) < 1. \end{cases} \tag{28}$$

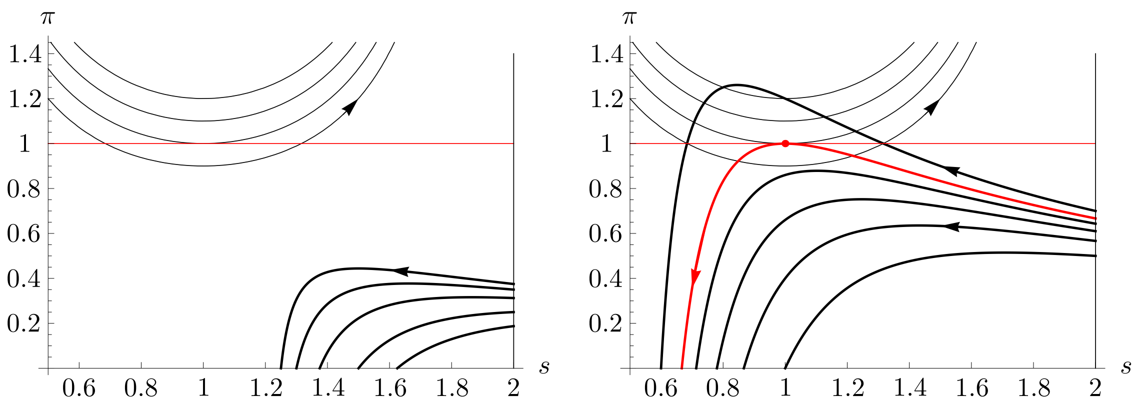

FIGURE 5. Left: Trajectories for $h = \bar{h} = 0.8$ (solid curves) and $h = 0$ (thin curves). Right: Trajectories for $h = \bar{h} = 1.5$ (solid curves) and $h = 0$ (thin curves), with the critical trajectory (red).

Since $\pi(t_1) < 1$ by Lemma 3, it follows from eq. (28) that the optimal path begins with $h(t_1) = \bar{h}$. Intuitively, since the initial stock equals its maximum level, $s(t_1) = 2$, there is no reason to begin with $h = 0$, and thus we begin with $h(t_1) = \bar{h}$. If time is relatively scarce, relative to the harvesting capacity, we continue with $h(t) = \bar{h}$ for all $t \in \mathscr{T}$; while if there is plenty of time, it is optimal to reduce harvesting in the meantime because else we had completed harvesting to early, and the terminal condition $\pi(T) = 0$ will not be met. More precisely, the optimal harvesting policy is as follows.

Lemma 5. *The optimal harvesting policy is given by*

$$h(t) = \bar{h} \qquad\qquad \text{if} \quad \bar{h} \le \bar{h}_c \tag{29}$$

$$h(t) = \begin{cases} \bar{h} & t_1 \le t < t_2, \\ 1 & t_2 \le t < t_3, \\ \bar{h} & t_3 \le t < T. \end{cases} \quad \text{if} \quad \bar{h} > \bar{h}_c, \tag{30}$$

with some critical harvesting capacity $\bar{h}_c > 1$ (depending on T).

Proof. See Appendix B or Hocking (1991). □

Figure 5 displays two types of trajectories: one for a small (left diagram) and one for a high (right diagram) harvesting capacity. The trajectories starting from $s(t_1) = 2$ reach the horizontal axis at time T, as the transversality condition requires $\pi(T) = 0$. If $\bar{h}$ is small, the trajectory does not reach the $\pi = 1$ line (see Figure 5, left); while if $\bar{h}$ is sufficiently large, it does. In fact, the proof of Lemma 5 shows that there exists a critical trajectory that touches the $\pi = 1$ line, and that this trajectory must feature $\bar{h} > 1$. For this reason, the critical harvesting capacity must exceed unity, *i. e.* $\bar{h}_c > 1$. Moreover, the critical harvesting capacity $\bar{h}_c$ depends inversely on the time horizon T.

Lemma 6. *Let $\psi : (1, 2] \to \mathbb{R}_+$ be defined by*

$$\bar{h} \mapsto \psi(\bar{h}) \equiv t_1 + \frac{1}{2-\bar{h}} \log\left(\frac{\bar{h}}{2(\bar{h}-1)^2}\right). \tag{31}$$

Then, given time T, the critical harvesting capacity $\bar{h}_c$ *is defined as the solution of $T = \psi(\bar{h})$,* i. e. $\bar{h}_c \equiv \psi^{-1}(T)$*. Equivalently, given some harvesting capacity $\bar{h}$, the* critical length of the harvesting period *is defined by $T_c \equiv \psi(\bar{h})$.*

The intuition for the optimal strategy given in Lemma 5 and the critical harvesting period given in Lemma 6 is as follows. (The proof of Lemma 6 is postponed until the end of the proof of the next lemma, see page 18.) In the case $T > T_c$, there is too much time for harvesting, implying that if the agent followed the critical path (the red path in the right diagram of Figure 5), they would have reached the $\pi = 0$–line too early. Thus, one might consider following a trajectory lying above the critical one, reaching the $\pi = 1$–line at some value $s > 1$. But then one has to switch to $h = 0$ following an upward-sloping trajectory (a thin path in the right diagram of Figure 5), implying that both the stock and the costate variable increase—and satisfying the terminal condition $\pi(T) = 0$ is impossible. For that reason the optimal policy is as follows: pursue the critical path up to $(s, \pi) = 1$, which is reached at time t_2; then, upon arrival at $(s, \pi) = 1$ reduce harvesting to $h = 1$, which, in view of eqs (25) and (27), renders both s and π to be constant, for 1 is the natural growth rate of the resource; finally, to complete the optimal path, resume maximal harvesting so as to arrive at $\pi = 0$ at time T.

4.3.1. *Case A: either $\bar{h} < 1$ or $1 < \bar{h} < 2$ and $T \le T_c$.* In this case, the maximal fishing effort is relatively low, $\bar{h} < \bar{h}_c = \psi^{-1}(T)$, so that $h(t) = \bar{h}$ can be maintained throughout. Then, the optimal fishing strategy is given by:

Lemma 7. *Let either $\bar{h} < 1$ or $1 < \bar{h} < 2$ and $T \le T_c$. Then the optimal fishing policy is given by*

$$h(t) = \bar{h}, \tag{32}$$

$$s(t) = \frac{2\left(\bar{h}-2\right)}{\bar{h}e^{\left(\bar{h}-2\right)(t-t_1)} - 2}, \tag{33}$$

$$\pi(t) = \frac{\bar{h}(s(T) - s(t))}{2s(t) - s(t)^2 - \bar{h}s(t)}, \tag{34}$$

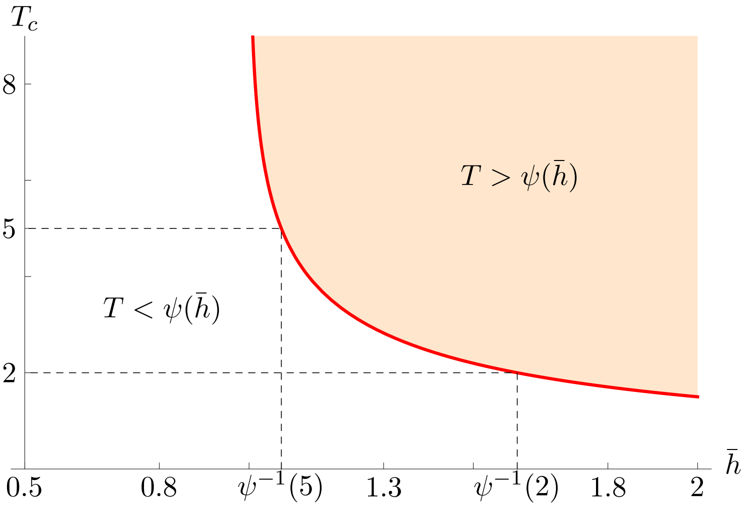

FIGURE 6. The critical value of T_c as function of $\bar{h}$.

for all $t \in \Delta$. The resulting maximised profit amounts to

$$J^*_{2A}(t_1) = \bar{h} \int_{t_1}^{T} s(t)\,\mathrm{d}t = \bar{h} \log \left(\frac{2e^{(2-\bar{h})(T-t_1)} - \bar{h}}{2-\bar{h}} \right). \tag{35}$$

Proof. We know from the proof of Lemma 5 that for all sub-critical cases $T < T_c$ (or $\bar{h} < \bar{h}_c$) defined in eq. (31), we have $h(t) = \bar{h}$ for all $t \in \Delta$. Substituting this, jointly with initial condition $s(t_1) = 2$ and the terminal condition $\pi(T) = 0$, into eqs (25)–(27) we obtain eqs (32)–(35). $\square$

Proof of Lemma 6. From eqs (33) and (34) we can calculate the critical time horizon T_c for which at some point in time t_c the trajectory goes through the point $(s(t_c), \pi(t_c)) = (1, 1)$. Using that information and evaluating eq. (34) at T_c yields $t_c = t_1 + \frac{1}{2-\bar{h}} \log \left(\frac{\bar{h}}{2(\bar{h}-1)} \right)$ and thus eq. (31). $\square$

Remark 2. For the limiting case when $\bar{h} \to 1$, the resulting profit amounts to $J^*_{2A}|_{\bar{h}=1}(t_1) = \log \left(2e^{T-t_1} - 1 \right)$.

The critical time horizon T_c is illustrated in Figure 6 (for $t_1 = 0$), and for $T = T_c$ we have:

Remark 3. In the critical case, *i. e.* when $T = T_c$, the optimal profit amounts to

$$J^c_{2A}(t_1) = 2\bar{h} \log \left(\frac{\bar{h}}{\bar{h}-1} \right). \tag{36}$$

4.3.2. *Case B:* $1 < \bar{h} < 2$ *and* $T > T_c$. In this case, the time available for harvesting $T - t_1$ is too long such that, given the maximal harvesting capacity $\bar{h}$, it is not optimal to harvest at the maximal rate all the time, as this would imply that $\pi = 0$ is reached before time T. Thus, harvesting cannot be maintained at rate $\bar{h}$ throughout, but must be reduced during some interval—and the optimal fishing strategy is as follows:

Lemma 8. *Let* $1 < \bar{h} < 2$ *and* $T > T_c$. *Then the optimal fishing policy is given by*

$$h(t) = \begin{cases} \bar{h} & t_1 \leq t < t_2, \\ 1 & t_2 \leq t < t_3, \\ \bar{h} & t_3 \leq t < T, \end{cases} \tag{37}$$

with switching times

$$t_2 = t_1 + \frac{\log\left(\frac{\bar{h}}{2(\bar{h}-1)}\right)}{2-\bar{h}} \quad \text{and} \quad t_3 = T - \frac{\log\left(\frac{1}{\bar{h}-1}\right)}{2-\bar{h}}.$$

The resulting profit is given by

$$J^*_{2B}(t_1) = T - t_1 + 2\bar{h}\log\left(\frac{\bar{h}}{\bar{h}-1}\right) - \frac{1}{2-\bar{h}}\log\left(\frac{\bar{h}}{2(\bar{h}-1)^2}\right), \tag{38}$$

Proof. That equation (37) is indeed the optimal fishing policy can be seen by noting that $\pi = 1$ is a singular level. Since we already know that $\pi(t_1) < 1$, it follows that $\dot{\pi} \geq 0$ at the time the singular level $\pi = 1$ is reached. If we have $\pi = 1$ for some time interval with positive length, then $\dot{\pi} = 0$ and hence we must have $s = 1 \Rightarrow \dot{s} = 0 \Rightarrow h = 1$ from eq. (25).

After completing the singular path we cannot have a path with $h = 0$. This can be seen as follows: $h = 0$ implies $\dot{s} > 0$, which in turn implies that, because $s = 1$ on the singular arc, $s > 1$ right after the singular arc. Jointly with $h = 0$ this in turn implies that $\dot{\pi} > 0$. Hence, we enter a path where both s and π are growing so that the transversality condition $\pi(T) = 0$ cannot be satisfied. We thus conclude that the optimal policy must proceed with $h = \bar{h}$ after completing the singular path—and thus the policy in eq. (37) is optimal.

The total length of the fishing sub-periods $[t_1, t_2)$ and $[t_3, T]$ amounts to $T_c - t_1$ and is thus given by eq. (31). Therefore fishing during these sub-periods brings about the same profit as in Case A, *i. e.* J^c_{2A} given by eq. (36); while during the time interval $[t_2, t_3)$ the resulting profit equals $h = 1$ times the length of the fishing period: $t_3 - t_2 = T - T_c$. So, we obtain

$$J^*_{2B} = J^c_{2A} + \int_{t_2}^{t_3} 1 \, \mathrm{d}t = J^c_{2A} + T - T_c.$$

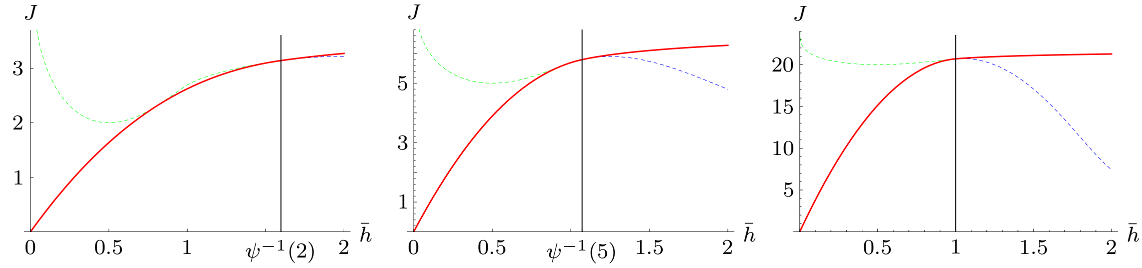

FIGURE 7. Profit in Case A and B for $T = 2$ (left), 5 (middle), 20 (right) with varying values of $\bar{h}$.

Finally, substituting the definition of T_c, eq. (31), yields eq. (38). □

In the limiting case where $T = T_c$, we have $t_2 = t_3$ and the central interval vanishes. More generally, since $\partial t_2/\partial \bar{h} < 0$ and $\partial t_3/\partial \bar{h} > 0$, the central interval increases with $\bar{h}$. The reason for this is that a higher harvesting capacity allows the agent to harvest more intensively in the beginning and at the end of the harvesting period, so that harvesting must be reduced in the central time interval. Yet, since $h(t) = 1$ is fixed for all $t \in [t_2, t_3)$, the only way to accomplish a lower catch in the central time interval is to extend this interval.

Remark 4. For the limiting case when $\bar{h} \to 1$, the resulting profit equals $J^*_{2B}|_{\bar{h}=1} = T - t_1 + \log(2)$; while for the case $\bar{h} \to 2$, the profit amounts to $J^*_{2B}|_{\bar{h}=2} = T - t_1 - \frac{3}{2} + \log(16)$. Finally, when the fishing capacity becomes unbounded, *i. e.* $\bar{h} \to \infty$, we obtain $J^*_{2B} = T - t_1 + 2$. Hence, we have $J^*_{2B}|_{\bar{h}=1} < J^*_{2B}|_{\bar{h}=2} < J^*_{2B}|_{\bar{h}=\infty}$, as expected.

As Remark 4 suggests, the maximised profit function J^*_2 is increasing in the capacity $\bar{h}$; this is depicted in Figure 7 for $T = 2$, 5 and 20. Therein, the vertical line represents the critical capacity $\bar{h}_c = \psi^{-1}(T)$. For values of $\bar{h} < \psi^{-1}(T)$ Case A applies; for values of $\bar{h} > \psi^{-1}(T)$, Case B. The critical values $\bar{h}_c = \psi^{-1}(T)$ can be gathered from eq. (31) *viz.* from Figure 6.

5. First Stage: Fixed travelling period

Having solved the harvesting problem, we now go back in time and solve the travelling problem. We begin our analysis with the simple, in our framework hypothetical, case of a fixed travelling period (sub-section 5), and then continue with acknowledging the subsequent harvesting period and endogenising the arrival time t_1 in sub-section 6. In this way, we are able to show which additional effects and which corresponding optimality conditions have to be added to the solution of the former problem to obtain the solution of the latter. We proceed in this successive manner for this allows us to spotlight the differences between the solution of the isolated travelling problem (6) and the solution of the travelling–and–harvesting problem (8).

Assume that the cost of travelling depends linearly on speed v and quadratically on acceleration a:

$$K(v, a) = cv + a^2. \tag{39}$$

Assuming $\rho = 0$ and $c = 1/10$, the resulting aggregated travelling cost amounts to

$$\int_0^{t_1} e^{-t\rho}\left(cv(t) + a(t)^2\right)\,\mathrm{d}t = \int_0^{t_1}\left(\frac{v(t)}{10} + a(t)^2\right)\mathrm{d}t \tag{40}$$

Acknowledging the constraints

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = a(t), \quad \dot{s}(t) = g(s(t)),$$

we obtain the Hamiltonian

$$\mathcal{H}_1 = -\frac{v(t)}{10} - a(t)^2 + \pi_2(t)a(t) + \pi_1(t)v(t).$$

For ease of tractability, we assume that there are no bounds on the control a—yet, we will drop this assumption in Appendix A. The familiar maximum principle then yields

$$\begin{aligned} x(t) =& \frac{t^2}{120}\left(30K_1 - 10K_2 t + t\right), & v(t) =& \frac{t}{40}\left(20K_1 - 10K_2 t + t\right), \\ \pi_1(t) =& K_2, & \pi_2(t) =& K_1 + \frac{t}{10}\left(1 - 10K_2\right), \end{aligned}$$

with K_1 and K_2 constants. Together with the boundary conditions $x(0) = v(0) = v(t_1) = 0$ and $x(t_1) = 1$, we obtain:

Proposition 1. *Given arrival time t_1, the optimal travelling policy is given by*

$$\begin{aligned} x(t) =& \frac{t^2(3t_1 - 2t)}{t_1^3}, \quad & v(t) =& \frac{6t(t_1 - t)}{t_1^3}, \quad & a(t) =& \frac{6(t_1 - 2t)}{t_1^3}, \\ \pi_1(t) =& \frac{24}{t_1^3} + \frac{1}{10}, & \pi_2(t) =& \frac{12(t_1 - 2t)}{t_1^3}, \end{aligned}$$

and the minimised objective function equals

$$J_1^*(t_1) = \int_0^{t_1}\left(a(t)^2 + \frac{v(t)}{10}\right)\mathrm{d}t = \frac{12}{t_1^3} + \frac{1}{10}. \tag{41}$$

Since J_1^* enters the objective function negatively, the value of the maximised Hamiltonian equals $\mathcal{H}_1^* = -dJ_1^*/dt_1 = \frac{36}{t_1^4}$. The acceleration of the vehicle and its resulting speed are depicted in Figure 8 for $t_1 = 1, \ldots, 5$.

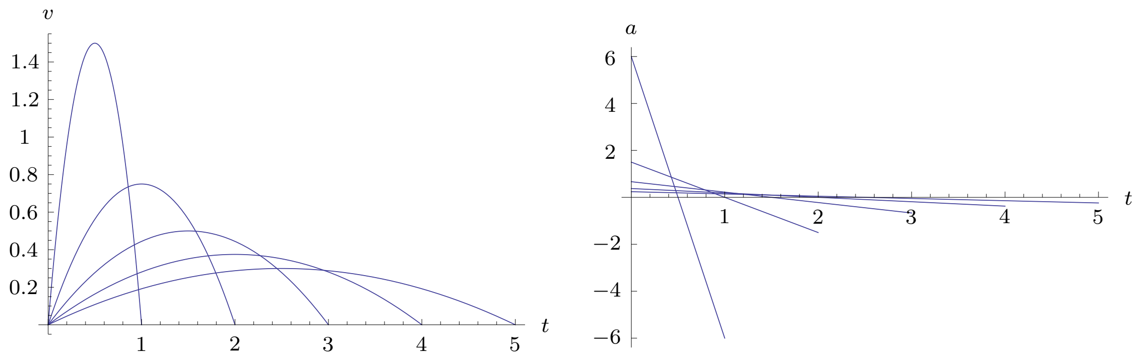

FIGURE 8. Speed and acceleration of the vehicle for $t_1 = 1, \ldots, 5$.

6. First Stage: Optimal travelling–and–harvesting policy

In Section 5 we assumed that t_1 is fixed. However, the agent may choose the length of the travelling period, and in this way the beginning of the harvesting period to maximise the profit. In order to determine the optimal policy for the travelling–and–harvesting problem, two different effects must be taken into account, and the associated conditions have to be added to those of the pure travelling decision.

First, the growth process of the resource during the travelling period must be acknowledged, and the corresponding necessary optimality condition need to be added to the canonical system:

$$\dot{s}(t) = g(s(t)) = \begin{cases} 2s(t) - s^2(t) & \text{logistic growth} \\ s(t) & \text{exponential growth,} \end{cases} \tag{42}$$

$$\dot{\pi}(t) = -\frac{\partial \mathcal{H}_1}{\partial s(t)} = -\pi(t) g'(s(t)) = \begin{cases} -2\pi(t)(1 - s(t)) & \text{logistic growth} \\ -\pi(t) & \text{exponential growth.} \end{cases} \tag{43}$$

Next, the terminal time t_1 and the endpoint s_1 of the travelling problem are free and may be chosen in an optimal way. While the arrival time t_1 determines the length of the harvesting period Δ, the endpoint s_1 determines the initial value of the growth process in the harvesting problem. Together, both effects determine the maximal value $J_2^*(s_1, t_1)$ of the harvesting period, which in turn represents the scrap value of the compound problem (8). However, the endpoint $s_1 = s(t_1)$ is fully determined by the arrival time, as the stock of the resource cannot be controlled before time t_1. For this reason, we do not have two, but only one transversality condition representing both effects: the direct effect of the arrival time on the length of the harvesting period Δ, and the effect of t_1 on the stock at the beginning of that period $s(t_1)$.

Hence, to derive a necessary condition for the optimal choice of the arrival time t_1, we first have to substitute the transversality condition (44), *i. e.* $s_1 = s(t_1) = s_0 e^{t_1}$, into J_2^*. Then, this value function, which may be viewed as a scrap value function of the travelling problem, can be written, with slight abuse of notation, as $J_2^*(t_1) \equiv J_2^*(s(t_1), t_1)$. Using this, the associated necessary condition for the free terminal time of the travelling problem t_1 reads as[17]

$$\mathcal{H}_1(s(t_1^*), a(t_1^*), \pi(t_1^*), t_1^*) + \frac{dJ_2^*(t_1^*)}{dt_1} = 0. \tag{44}$$

With the help of condition (44) we are now able to calculate the optimal travelling–and–harvesting policy. We do this for both growth functions specified above.

6.1. Optimal Travelling–and–harvesting policy for exponential growth. Acknowledging the transversality conditions, the following conditions have to be added

$$s(t) = s_0 e^t, \qquad \pi(t) = \left(\frac{1}{\bar{h}}\right)^{\frac{1}{\bar{h}-1}} e^{T-t}. \tag{45}$$

We next show that t_1 must not be smaller than the switching time τ, so that harvesting begins immediately upon arrival. Intuitively, this is because a premature arrival is costly without yielding any additional profit, as we initially have $h(t) = 0$ in Case B. So Case A applies, and the maximised value function of the harvesting problem $J_{2A}^*(s_1, t_1)$ is given by eq. (21).

Proposition 2. *In the optimal travelling–and–harvesting policy, the optimal arrival time t_1^* succeeds time τ,* i. e. *Case A applies. Hence, the optimal harvesting policy is characterised in Lemma 1, and the resulting profit from travelling–and–harvesting policy is given by*

$$V^* \equiv J_{2A}(t_1^*) - J_1(t_1^*) = s_0 e^{t_1^*} \frac{\bar{h}}{\bar{h}-1}\left(1 - e^{(\bar{h}-1)(t_1^*-T)}\right) - \left(\frac{12}{(t_1^*)^3} + \frac{1}{10}\right), \tag{46}$$

where t_1^* is a function of $\bar{h}$ and T, implicitly defined as the solution of

$$\frac{36}{t_1^4} - s_0 e^{t_1} \frac{\bar{h}}{\bar{h}-1}\left(\bar{h} e^{(\bar{h}-1)(t_1-T)} - 1\right) = 0. \tag{47}$$

Before we prove Proposition 2, we derive a lemma characterising J_{2A}^*.

Lemma 9. *The derivative of the maximised value of the harvesting problem is determined by the switching point τ:*

$$\frac{dJ_{2A}^*(t_1)}{dt_1} \gtreqqless 0 \quad \Leftrightarrow \quad T - t_1 \gtreqqless \delta \equiv \frac{\log(\bar{h})}{\bar{h}-1} \quad \Leftrightarrow \quad \tau \gtreqqless \delta \equiv t_1.$$

[17] Condition (44) represents a modification of the usual necessary condition for the free terminal time, as provided, for example, by Léonard and Long (1992, Theorem 7.6.1).

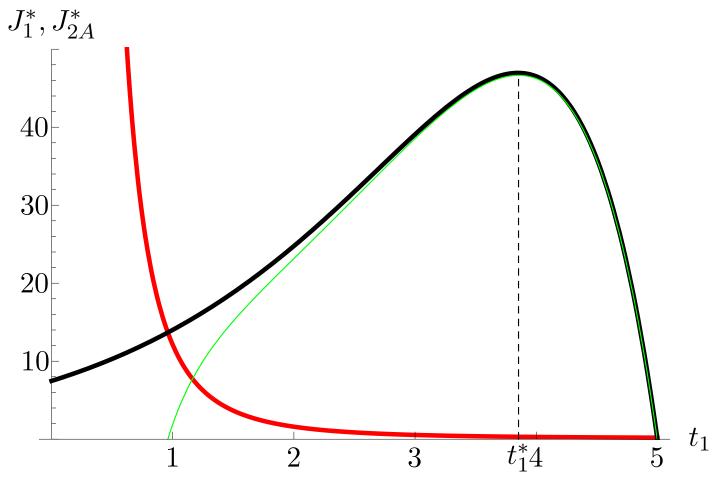

FIGURE 9. Exponential growth: value function J_{2A}^* (black), cost function J_1^* (red), and profit function $-J_1^* + J_{2A}^*$ (green), for $s_0 = 1$, $T = 5$ and $\bar{h} = 3/4$, yielding the optimal arrival time $t_1^* = 3.8539$ and the net profit $J_{2A}(t_1^*) - J_1(t_1^*) = 46.6488$.

Proof. After substitution of $s(t_1) = s_0 e^{t_1}$, we obtain the derivative of the value function $J_{2A}^*(t_1) \equiv J_{2A}^*(s(t_1), t_1)$:

$$\frac{dJ_{2A}^*(t_1)}{dt_1} = -s_0 e^{t_1} \frac{\bar{h}}{\bar{h} - 1} \left(\bar{h} e^{\left(\bar{h}-1\right)(t_1 - T)} - 1 \right).$$

It is then straightforward to show that the sign of this derivative depends on whether the switching point τ is before or after the arrival time t_1. □

Since δ is a decreasing function of $\bar{h}$, the derivative of dJ_{2A}^* is positive for large, and negative for small values of $\bar{h}$. If the harvesting capacity, when compared with the length of the harvesting period $\Delta \equiv T - t_1$, is large, a later arrival time increases the yield from the harvesting period because it gives the resource more time to grow; at the same time, the harvesting capacity is large enough so as to harvest high volumes in a shorter time interval. Consequently, in this case the agent may wish to postpone the arrival time. However, when the harvesting capacity is relatively low, postponing the start of the harvesting activity is unattractive, as the agent will be unable to benefit from the higher stock due to the constraint on the harvesting capacity. Hence, in the absence of travelling costs the optimal arrival time will be equal to $t_1^* = \tau \equiv T - \delta$. This arrival time balances the benefits from an earlier and a later arrival.[18]

Proof of Proposition 2. Using the maximised Hamiltonian of sub-section 5, $\mathcal{H}_1^* = 36/t_1^4$, the transversality condition (44) gives eq. (47). Since $\mathcal{H}_1^*$ is positive, the

[18]Indeed, Case A and Case B coincide for $\tau = t_1$.

derivative $dJ_{2A}^*(t_1)/dt_1$ must be negative in order for (47) to have a solution t_1^*. By Lemma 9, $dJ_{2A}^*(t_1)/dt_1$ is negative if, and only if, the switching point τ is before the arrival time: $t_1^* > T - \delta \equiv \tau$ implying that Case A applies. $\square$

Hence, the optimal arrival time t_1^* is chosen so that the harvesting activity begins immediately upon arrival. In other words, Case A applies, *i. e.* $T - t_1 < \delta$, and the agent begins with harvesting at the maximum rate immediately at time t_1. This is because an early arrival is associated with higher travelling cost, so that this should be avoided. (The functions $J_{2A}(t_1)$ and $J_1^*(t_1)$ are depicted in Figure 9.)

It also follows from condition (47) that in the optimal travelling–and–harvesting policy, the length of the harvesting period $\Delta \equiv T - t_1$ is smaller than the harvesting period the agent would have chosen in the absence of the need for travelling (assuming $T > \delta$). Consequently, in the presence of travelling cost the optimal arrival is postponed, and the harvesting process begins at some later time, compared with the case of the absence of the need to travel.

6.2. Optimal travelling–and–harvesting policy for logistic growth. By assumption $s(t_1)$ is fixed at $s_1 = 2$, so that we can choose only t_1 without affecting s_1. With logistic growth we have to consider both Case A and Case B.

6.2.1. *Case A: either $\bar{h} < 1$ or $1 < \bar{h} < 2$ and $T \leq T_c$.* In this case, the optimal harvesting policy is characterised by Lemma 7), and the functions $J_1^*(t_1)$ and $J_{2A}^*(t_1)$ are given by eqs (41) and (35) respectively. Moreover, the time derivative of $J_{2A}^*(t_1)$ equals

$$\frac{dJ_{2A}^*(t_1)}{dt_1} = -\frac{2\left(\bar{h}-2\right)\bar{h}}{\bar{h}e^{\left(\bar{h}-2\right)(T-t_1)} - 2}.$$

This derivative is negative since, due to $\bar{h} < 2$, the numerator and the denominator are both negative. Then, the optimal arrival time is determined, again, by eq. (44). Hence, the optimal travelling–and–harvesting policy is characterised as follows:

Proposition 3. *Let either $\bar{h} < 1$ or $1 < \bar{h} < 2$ and $T \leq T_c$. Then, the optimal harvesting policy is $h(t) = \bar{h}$ for all $t \in \Delta$, and the resulting profit from travelling–and–harvesting policy is given by*

$$V^*(T) \equiv J_{2A}(t_1^*) - J_1(t_1^*) = \bar{h}\log\left(\frac{2e^{(2-\bar{h})(T-t_1)} - \bar{h}}{2-\bar{h}}\right) - \left(\frac{12}{(t_1^*)^3} + \frac{1}{10}\right), \tag{48}$$

where t_1^* is a function of $\bar{h}$ and T, implicitly defined as the solution of

$$\frac{36}{t_1^4} - \frac{2\left(\bar{h}-2\right)\bar{h}}{\bar{h}e^{\left(\bar{h}-2\right)(T-t_1)} - 2} = 0. \tag{49}$$

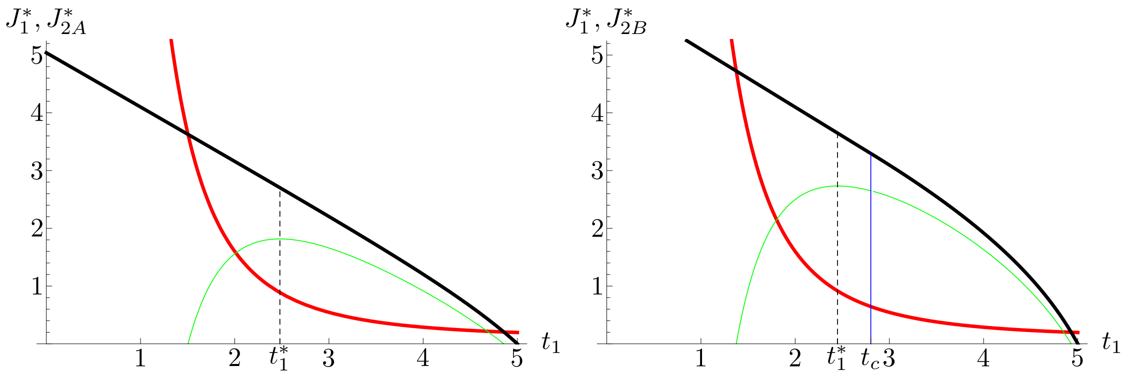

FIGURE 10. Logistic growth: value function J_2^* (black), cost function J_1^* (red), and profit function $-J_1^* + J_2^*$ (green), for $T = 5$. Case A (left): $\bar{h} = 3/4$, optimal arrival time $t_1^* \approx 2.4793$. Case B (right): $\bar{h} = 3/2$, optimal arrival time $t_1^* = \sqrt{6} \approx 2.4495$ with critical arrival time $t_c = 5 - 2\log(3) \approx 2.8028$ (blue).

The functions $J_1^*(t_1)$ and $J_{2A}^*(t_1)$ are depicted in Figure 10 for a low (left diagram) and a high (right diagram) harvesting capacity. Setting $\bar{h} = 3/4$, the optimal arrival time equals $t_1^* \approx 2.4793$ yielding a net profit equal to $J_{2A}(t_1^*) - J_1(t_1^*) \approx 1.8161$. Carefully observe that Case A actually materialises (for $T = 5$), compare the second diagram in Figure 7

6.2.2. *Case B:* $1 < \bar{h} < 2$ *and* $T > T_c$. In this case, the harvesting capacity $\bar{h}$ exceeds the critical value $\bar{h}_c$, and the resulting optimal harvesting policy is characterised in Lemma 8. Hence, the optimal travelling–and–harvesting policy is characterised as follows:

Proposition 4. *Let* $1 < \bar{h} < 2$ *and* $T > T_c$. *Then, the optimal harvesting policy is characterised by Lemma 8, and the associated profit is given by eq.* (38). *The resulting profit from the optimal travelling–and–harvesting policy is given by*

$$\begin{aligned} V^*(T) &\equiv J_{2B}(t_1^*) - J_1(t_1^*) \\ &= T - t_1^* + 2\bar{h}\log\left(\frac{\bar{h}}{\bar{h}-1}\right) - \frac{1}{2-\bar{h}}\log\left(\frac{\bar{h}}{2(\bar{h}-1)^2}\right) - \left(\frac{12}{(t_1^*)^3} + \frac{1}{10}\right), \end{aligned} \tag{50}$$

where t_1^* *implicitly defined as the solution of* $36/t_1^4 = 1$, *the single positive and real root of which is* $t_1^* = \sqrt{6}$.

Apparently, $J_{2B}^*(t_1)$ is linear in the arrival time, so that $dJ_{2B}^*(t_1)/dt_1 = -1$. This scenario is depicted for $\bar{h} = 3/2$ and ($T = 5$) in the right part of Figure 10, where the value function $J_{2B}^*(t_1)$ is linear for all arrival times $t_1 < t_c$. (Recall that for these parameter values, Case B results, which is illustrated in the second diagram in Figure 7). With these parameters, the optimal solution is given by $t_1^* \approx 2.4495$ yielding a net profit of $J_{2B}(t_1^*) - J_1(t_1^*) \approx 2.7326$.[19]

7. Conclusion

In this paper we contribute to the theory of spatial resource economics. We explicitly take into account the fact that in many real-world situations the agent has to travel to the location of the resource before being able to begin with harvesting. Although some papers in the literature acknowledge the requirement of an agent to travel to the resource (*e. g.* Behringer and Upmann, 2014; Belyakov et al., 2015), the approach of this paper is different in that the resource cannot be harvested in an *en passant* manner, but the agent has to stop at the location of the resource in order to harvest. As a consequence, the travelling problem and the subsequent harvesting problem are linked by the choice of the speed of travelling and, thus, by the resulting arrival time, as the latter determines both, the start of the harvesting period and the initial value of the size of the stock. The crucial effect is that while a longer travelling period delays the harvesting activity, which is in principle an unwelcome effect, a later arrival also gives the resource more time to grow unimpaired. In this way, the travelling decision determines both the length of the harvesting period and the initial conditions for the harvesting activity. The two sub-problems are therefore linked by the spatiotemporal dimension, rendering the arrival time a crucial decision of the optimal harvesting policy.

We are able to fully characterize the control programme for the composed travelling–and–harvesting problem, employing recent tools for two-phase dynamic optimization problems. In particular, we derive the optimal value functions for both the travelling and the harvesting sub-problem, and then provide the additional conditions required to acknowledge the link between both problems. This procedure allows us to thoroughly characterise the optimal travelling–and–harvesting policy and the resulting optimal yield for the management of a remote resource.

We derive this optimal policy for two different stipulated growth processes of the stock of the resource: exponential and logistic growth. Since both growth functions bring about qualitatively similar results, we conclude that the effects we identified are quite robust in this regard. In order to investigate the robustness of our results further, we consider the case of a positive discount rate and bounds

[19] Had we chosen some later starting time $t_1 > t_c \equiv T - T_c = 5 - 2\log(3) \approx 2.8028$, then Case A became relevant as the fisher had less than the required minimal time for fishing in Case B, $T_c = 2\log(3) \approx 2.1972$.

on acceleration in the Appendix A, where we show that a positive discount rate induces a shift of acceleration and hence speed costs towards the future, whereas bounds on acceleration hamper that effect. Again, the crucial link between the travelling and the harvesting problem persists.

Overall, we have demonstrated that acknowledging the spatial dimension in the classical problem of managing a renewable resource can lead to interesting and economically relevant, yet still analytically tractable, changes that even allow for the introduction of more realistic features, such as periods of travelling and harvesting and their associated economic costs. This extension, besides contributing to the call of introducing a spatial dimension and thus to enhance the realism of the model, allows for an even more precise extension of the theory into a realm where space implies that the agent also faces a transportation problem that is temporarily and spatially linked to the resource extraction problem.

Appendix A. Robustness analysis: a positive discount rate and bounds on the acceleration

We here explore the robustness of the optimal travelling policy. Maintaining our specification of the travelling cost (39) used in section 5, we assume that K depends linearly on speed v and quadratically on acceleration a with $c = 1/10$. For illustrative purposes, we now set $\rho = 1/20$ and $t_1 = 40$. Also, we assume that acceleration is bounded to $\mathscr{A} = [\underline{a}, \overline{a}] = [-1, +1]$.

With this specification the objective function is given by

$$J_1(a, 40) = \int_0^{40} e^{-t/20} \left(\frac{v(t)}{10} + a(t)^2 \right) \mathrm{d}t.$$

Acknowledging the system of differential equations (1) governing the movement of the agent

$$\dot{x}(t) = v(t) \quad \text{and} \quad \dot{v}(t) = a(t) \quad \forall t \in \mathscr{T},$$

the Hamiltonian is given by

$$\mathcal{H}_1 = -K(v(t), a(t)) + \psi_1(t)v(t) + \psi_2(t)a(t) = -\frac{v(t)}{10} - a(t)^2 + \psi_1(t)v(t) + \psi_2(t)a(t),$$

and the Lagrangean reflecting the restriction $\underline{u} \leq u \leq \overline{u}$ by

$$\mathcal{L} = -\frac{v(t)}{10} - a(t)^2 + \pi_1(t)v(t) + \pi_2(t)a(t) + \lambda_1(t)(a(t) + 1) + \lambda_2(t)(1 - a(t)).$$

Subsequently, the Hamiltonian and the costate variables are defined in current values, so that the necessary conditions are modified accordingly:

$$-2a(t) + \pi_2(t) + \lambda_1(t) - \lambda_2(t) = 0 \quad \Leftrightarrow \quad a(t) = \frac{1}{2} \left(\pi_2(t) + \lambda_1(t) - \lambda_2(t) \right). \quad \text{(A.1)}$$

In addition, we have the necessary conditions

$$\dot{\pi}_1(t) = -\frac{\partial \mathcal{L}}{\partial x(t)} + \rho\pi_1(t) = \frac{\pi_1(t)}{20}, \tag{A.2}$$

$$\dot{\pi}_2(t) = -\frac{\partial \mathcal{L}}{\partial v(t)} + \rho\pi_2(t) = -\pi_1(t) + \frac{\pi_2(t)}{20} + \frac{1}{10}. \tag{A.3}$$

A.1. Analysis of the unbounded solution. In this case we have $\lambda_1(t) = 0 = \lambda_2(t)$, and equations (1) and (A.1) simplify to

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = \frac{\pi_2(t)}{2} \tag{A.4}$$

jointly with equations (A.2) and (A.3). Using the, yet unspecified, initial values $\pi_1(0) = m_1$ and $\pi_2(0) = c_1$, we obtain

$$\pi_1(t) = m_1 e^{t/20}, \qquad \pi_2(t) = e^{t/20}(c_1 - m_1 t + 2) - 2. \tag{A.5}$$

Substituting eq. (A.5) into (A.1) yields the system

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = \frac{1}{2}\left(e^{t/20}(c_1 - m_1 t + 2) - 2\right), \quad x(0) = 0, \quad v(0) = 0.$$

Using the initial values and solving the resulting initial value problem, we obtain

$$\begin{aligned}
x(t) &= \frac{1}{2}\Big(20c_1\left(-t + 20e^{t/20} - 20\right) - 400m_1 t \\
&\qquad -400e^{t/20}(m_1(t-40) - 2) - 16000m_1 - t^2 - 40t - 800\Big), \\
v(t) &= 10c_1\left(e^{t/20} - 1\right) - 10e^{t/20}(m_1(t-20) - 2) - 200m_1 - t - 20. \\
\pi_1(t) &= m_1 e^{t/20}, \\
\pi_2(t) &= e^{t/20}(c_1 - m_1 t + 2) - 2.
\end{aligned}$$

Finally, using the terminal conditions $x(t_1) = 300$ and $v(t_1) = 0$ to determine the constants, we obtain

$$c_1 = \frac{-25 + 35e^2 - 4e^4}{2\left(1 - 6e^2 + e^4\right)}, \quad m_1 = \frac{13 + 3e^2}{40\left(1 - 6e^2 + e^4\right)}.$$

Thus, the optimal control and maximised objective function are given by

$$\begin{aligned}
a(t) &= \frac{e^{\frac{t}{20}+2}(220 - 3t) - e^{t/20}(13t + 420) - 80e^4 + 480e^2 - 80}{80\left(1 - 6e^2 + e^4\right)}, \\
J_1^* &= \frac{80 - 1725e^2 + 925e^4 - 80e^6}{4\left(e^2 - 6e^4 + e^6\right)} \approx 16.7095,
\end{aligned}$$

respectively.

A.2. Analysis of the bounded solution. Now, assume that there are bounds on the control: $a(t) \in [\underline{a}, \overline{a}] \equiv [-1, 1]$. As we can see from Figure 11, the unbounded solution (blue case) hits the lower bound, at time $t \approx 34.2818$. Since the upper bound $\overline{a} = 1$ is not binding, it suffices to consider the former only: We must have $a(t) = \underline{a} = -1$ for all t in the final interval $(\xi, t_1]$. Apparently, we must choose some $\xi < 34.2818$, for if $\xi = 34.2818$ the remaining time would only suffice to guarantee the terminal condition $v(t_1) = 0$, if we were able to set $a < \underline{a}$. Thus, during the final time interval $(\xi, t_1]$, the solution must satisfy

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = -1, \quad x(40) = 300, \quad v(40) = 0,$$

the solution of which is

$$x(t) = \frac{1}{2}\left(-t^2 + 80t - 1000\right), \quad v(t) = 40 - t, \quad a(t) = -1.$$

We now have to calculate the optimal switching point ξ, which must be determined so that the following boundary conditions (for the first interval) are met:

$$x(0) = 0, \quad v(0) = 0, \quad x(\xi) = \frac{1}{2}\left(-\xi^2 + 80\xi - 1000\right), \quad v(\xi) = 40 - \xi.$$

Together with the optimality conditions for the unbounded problem, eqs (A.2), (A.3) and (A.4), this yields the system

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = \frac{\pi_2(t)}{2}, \tag{A.6}$$

$$\dot{\pi}_1(t) = \frac{\pi_1(t)}{20}, \qquad \dot{\pi}_2(t) = -\pi_1(t) + \frac{\pi_2(t)}{20} + \frac{1}{10}, \tag{A.7}$$

$$x(0) = 0, \qquad v(0) = 0, \tag{A.8}$$

$$x(\xi) = \frac{1}{2}\left(-\xi^2 + 80\xi - 1000\right), \qquad v(\xi) = 40 - \xi, \tag{A.9}$$

the solution of which gives the optimal travelling policy in the interval $[0, \xi]$:

$$\begin{aligned} a(t) = {} & \frac{1}{-e^{\xi/20}\left(\xi^2 + 800\right) + 400e^{\xi/10} + 400} \times \\ & \Big(e^{\frac{t+\xi}{20}}\left(t(65 - 2\xi) + \xi(2\xi - 105) + 2100\right) - 5e^{t/20}(13t + 420) \\ & + e^{\xi/20}\left(\xi^2 + 800\right) - 400e^{\xi/10} - 400\Big), \end{aligned}$$

which reaches the lower bound $\underline{a} = -1$ at time $\xi \approx 29.5984$. Using this value and composing both parts, we obtain the optimal control:

$$a(t) = \begin{cases} \frac{1}{2}\left(e^{t/20}(3.09497 - 0.104565t) - 2\right) & 0 \le t \le \xi \\ -1 & \xi < t \le t_1 \end{cases},$$

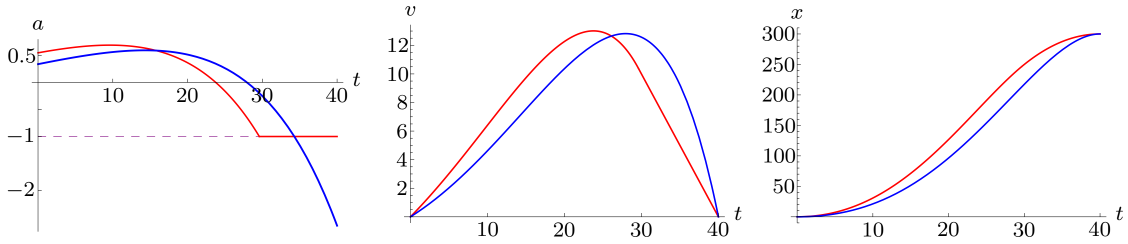

FIGURE 11. Optimal acceleration, speed and position with (red) and without (blue) bounds on acceleration.

and the resulting value of the minimal travelling cost amounts to $\overline{J}_1^* \approx 18.4648$. The optimal solution is illustrated by the red trajectories in Figure 11. Compared to $J_1^* \approx 16.7095$ for the case of an unbounded control, the presence of the bound on acceleration results in an increase in the minimal travelling cost.

We may also compare our result with the case of a zero discount rate, explored in Section 6. Applying the specification $t_1 = 40$, $x(t_1) = 300$ and $\rho = 0$, we obtain $J_1^*|_{\rho=0} = 375/8 = 46.875$. Apparently, with discounting parts of the cost vanish, so that the present value of the cost without discounting exceed those with discounting. Also, as can be seen from Figure 11, discounting makes the agent initially move more slowly and speed up later so that part of the travelling cost is shifted to the future. In case of bounds on the control, such cost shifting becomes limited so that some part of the travelling cost must be incurred earlier.

We have thus demonstrated that our qualitative results for the optimal travelling–and–harvesting policy need to be modified—in an intuitive way—in response to the introduction of either bounds on acceleration or a positive discount rate. In particular, the introduction of a positive discount rate makes the agent postpone part of the costly travelling activity to the future: Whereas the original path is decreasing linearly leading to a symmetric and concave velocity curve, the optimal acceleration under discounting is concave, increasing first and then decreasing. This reflects the fact that present acceleration and velocity become more costly compared to future ones. As a consequence, travelling costs are shifted towards the future. In case that acceleration is bounded from below, slowing down at the end of the travelling period is constrained, implying a higher (and therefore costlier) acceleration at the start and subsequently a sooner reduction of acceleration, since otherwise the destination could not be reached with zero speed. The velocity curves and the optimal position curves reflect these optimal acceleration patterns accordingly.

Appendix B. Proof of Lemma 5

Proof. Since the Hamiltonian is autonomous, it is constant along the optimal trajectory.[20] We can therefore characterise the trajectories in the (s, π) plane for $h = 0$ and for $h = \bar{h}$. Let K denote the level of the Hamiltonian, then the optimal trajectories are characterised by the equations

$$\pi(t) = \frac{K}{2s(t) - s^2(t)} \quad \text{and} \quad \pi(t) = \frac{K - s(t)\bar{h}}{2s(t) - s^2(t) - s(t)\bar{h}} \tag{B.10}$$

for the cases $h = 0$ and $h = \bar{h}$, respectively. The $h = 0$–trajectories have their minima at $s = 1$, and the trajectories with $h = \bar{h}$ attain their maxima along the curve

$$\pi(t) = \frac{-\bar{h}}{2 - 2s(t) - \bar{h}} \quad \text{for } s > 1 - \frac{1}{2}\bar{h}. \tag{B.11}$$

Both types of trajectories are depicted in Figure 5 for a low (left diagram) and a high (right diagram) harvesting capacity. The trajectories starting from $s(t_1) = 2$ reach the horizontal axis at time T, *i. e.* $\pi(T) = 0$. Those trajectories with $\bar{h} < 1$ cross the horizontal axis at a point to the right of $2 - \bar{h}$, that is $s(T) > 2 - \bar{h}$. If $\bar{h}$ is sufficiently small, the trajectory does not reach the $\pi = 1$ line (see Figure 5, left). Since the locus of maxima crosses the point $(1, 1)$, see eq. (B.10), the critical trajectory is that one which achieves its maximum at this point (see Figure 5, right). Because the trajectories do not cross the horizontal axis to the left of $2 - \bar{h}$, the critical trajectory must feature $\bar{h} > 1$. It thus follows that the critical harvesting capacity exceeds unity, $\bar{h}_c > 1$. □

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[20] See, for example, Intriligator (1971, p. 355).

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