Harvesting a Remote Renewable Resource

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Abstract

In standard models of spatial harvesting, the resource is distributed over the complete domain and the agent is able to control the harvesting activity everywhere all the time. In some cases though, it is more realistic to assume that the resource is located at a single point in space and that the agent is required to travel there in order to be able to harvest. In these cases, the agent faces a combined travelling–and–harvesting problem. We scrutinize this type of a two-stage optimal control problem, and illuminate the interdependences between the solution of travelling and that of the harvesting sub-problem. Since the model is parsimoniously parameterised, we are able to analytically characterise the optimal policy of the complete travelling–and–harvesting problem, even in the presence of bounds on the harvesting capacity. In an appendix, we show how a bound on the control of travelling, *i. e.* on acceleration, as well as a positive discount rate affect the solution of the problem.

Keywords: Optimal travelling–and–harvesting decision; spatial renewable resource; optimal control; two-stage control problem

JEL classification: Q20, Q22, C61

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1. Introduction

In the management of renewable natural resources, the spatial dimension has recently attracted substantial attention. Frequently, the focus of this literature is on finding out how much effort is required to harvest a resource, such as fish or game, within a certain domain when that resource is moving. In this paper, we reverse this view: we consider the case where an agent is required to move within the domain in order to be able to harvest an immobile resource. Since travelling is a pre-requisite of harvesting, there is spatio-temporal interdependence of both policies; this is the focus of our paper.

The management of renewable natural resources has been a central issue in economics since Gordon (1954) and Smith (1968) initially advanced this topic. In their monographs, Conrad and Clark (1987), Conrad (2010) and Clark (2010) nicely demonstrate how optimal control theory may constructively contribute to the management of renewable resources, and fisheries in particular. Subsequently, these textbook models have been extended and generalised in various respects. For example, Fan and Wang (1998) generalise the optimal harvesting policy of an autonomous harvesting problem with logistic growth (see, for example, Clark, 2010) to a non-autonomous case with periodic coefficients; Liski et al. (2001), accounting for costly changes of the harvesting rate, explore the effects of increasing returns to scale for a standard fishery management model; and Ainseba et al. (2003), Feichtinger et al. (2003), Hritonenko and Yatsenko (2006), Tahvonen (2008, 2009a,b), Li and Yakubu (2012), Skonhoft et al. (2012), Quaas et al. (2013), Tahvonen et al. (2013) and Belyakov and Veliov (2014) investigate harvesting of age-structured populations.

While the above-mentioned work takes into account the temporal and the bioeconomic dimension, the spatial dimension—though already present in the literature of theoretical biology and applied mathematics—became of interest to economists relatively late: only in 1999, Sanchirico and Wilen bring the spatial dimension to the attention of resource economists. In their seminal paper, Sanchirico and Wilen (1999) generalize the fundamental open-access models of Gordon (1954) and Smith (1968) by the spatial direction: they set up a bioeconomic model with a finite number of resource patches within which the biomass migrates, resulting in time dependent changes in the allocation of effort between these patches. In this way, the authors integrate within- and between-patch biological and economic forces and demonstrate how these effects determine the process of bioeconomic convergence over space and time.[1]

Following Sanchirico and Wilen (1999), the early models in spatial resource economics feature discrete patches, where the stock evolves according to an ordinary differential

[1]In a subsequent paper Sanchirico and Wilen (2005) utilize the model from their 1999 paper to characterise the spatially differentiated landings and effort taxes suitable to implement a first-best allocation.

equation at each location; migration of the biomass is then modelled as entry and exit of the biomass from one location to the other. An alternative approach models the migration and the spread of the biomass as diffusion described by partial differential equations.[2] Notable contributions are Cañada et al. (1998), Montero (2000, 2001), Neubert (2003), Bai and Wang (2005), Brock and Xepapadeas (2008, 2010), Ding and Lenhart (2009), Joshi et al. (2009), Bressan et al. (2013), Uecker and Upmann (2016).[3]

In both strands of the literature, it is the biomass which is mobile while the movement of the agent remains unrecognised. In many instances an immobile agent represents a reasonable simplification, as the effect of the agent's movement can be neglected without loosing much realism (*e. g.* coastal fixed–net fishery or shooting game)—but in other cases it is not. For example, in fruit and mushroom harvesting (in distant, vast wilderness such as Canada or Kamchatka), in forestry and in extensive agriculture *etc.*, it is the agent who is moving in order to access the resource that is located at some fixed known patch. Here, the travelling costs are often significant due to either large distances or because of a lack of infrastructure. Sometimes there is only a short window in which the spatially distributed fruits can be gathered, as, for example, grape harvesting needs to be done within the last few days before the first night frost emerges (late harvests). Another example which has recently attracted much attention is algae harvesting; algae is a rapidly growing renewable resource, which may be used in the production of human and animal food, cosmetics, pharmaceuticals, chemicals, plastics, and biofuel.[4] Algae is frequently located at distant patches and is hard to monitor; in particular, algae farms have recently been automated and located more and more off-shore so that travelling times become increasingly relevant for optimal harvesting decisions. Finally, the same structural argument, though with a "negative sign", applies to the containment of sprawling environmental damage, where the agent is required to approach the source of the environmental damage before fighting it.

In this paper, we build on these observations and analyse the optimal behaviour of an agent who is required to travel in order to harvest a remote resource. In contrast to the standard approach in spatial resource economics with a mobile resource, we reverse the abilities of movement: When the resource is immobile, the agent first has to get to its location before being able to harvest. Consequently, any admissible policy consists of

[2]The basic population models with diffusion are presented in, for example, Aniţa (2000, sec. 1.2), Okubo and Levin (2001), Murray (2003) and the references therein.

[3]For a recent and extensive overview of the literature of spatial-dynamic systems in resource economics, see Kroetz and Sanchirico (2015).

[4]For contemporary literature on the possibilities of algal biofuel production, see, for example, Shurin et al. (2013); Zhu et al. (2017); Ummalyma et al. (2017); Chung et al. (2017); Hoffman et al. (2017); Chu (2017); for a paper investigating algae growth control from a pollution point of view, see Yoshioka and Yaegashi (2018).

two disjunct time intervals: a travelling interval and a harvesting interval. This reversal of movement enhances the realism in modelling natural resource management when the resource is rather immobile and located at some place that is remote or small compared to the overall domain and is thus costly to access. In this way, this paper complements the contemporary literature on spatial resource economics by focusing on the interdependence of the travelling and the harvesting decision.

Few papers consider a travelling–and–harvesting problem of the agent in a spatial domain. Notable examples are Robinson et al. (2008), Behringer and Upmann (2014), Belyakov et al. (2015, 2017) and Zelikin et al. (2017), who consider an immobile resource located at known locations.[5] Except for Robinson et al. (2008), all of these authors analyse a resource that is continuously distributed on the periphery of a circle and an agent who leaves for a round trip, returning home after each turn. In those models the agent does not need to stop in order to gather the resource, but is able to do this *en passant*. While in the model of Behringer and Upmann (2014) the harvesting activity does not cost any time, over and above the time of travelling, in Belyakov et al. (2015, 2017) the maximal harvesting activity is inversely proportional to velocity.[6] Still, in both cases, the travelling and the harvesting activity occur simultaneously and may even be identified with each other. This is opposite to our approach here, where travelling and harvesting are mutually exclusive, rival activities: the more time is spent on travelling, the less time is left for harvesting, and *vice versa*.

Robinson et al. (2008) provide a timber gathering model which has parallels with our paper. Their model also treats discrete resource patches and assumes that travelling and harvesting are exclusive, as we do in this paper. While we derive analytical solutions for the harvesting and the travelling processes themselves, each governed by an independent control, in Robinson et al. (2008) the decisions about travelling speed and the amount harvested are connected via a gathering-specific cost function; those authors formalize the idea that the travelling time, and hence the travelling cost, increases in the total amount already harvested, as this harvest has to be carried back to a nearby market.[7] In our approach, travelling and harvesting are distinct activities (in contrast to the "search

[5]For a review of modelling spatial resource harvesting see, for example, Conrad and Smith (2012) or Albers and Robinson (2013).

[6]Models of this type are sometimes referred to as "search models", as a higher speed of travelling renders the agent incapable of descrying much of the resource (see also Robinson et al., 2002).

[7]In our setting, there is no need to carry the harvest back to some central market base. This can be motivated by immediate consumption or by multiple markets that are located close to the resource patches. When extending the analysis to multiple patches this comes at the analytical advantage that patches can be treated symmetrically. For a model that investigates the effects of endogenous market demands in a spatial harvesting setting see Aniţa et al. (2019).

models", see fn. 6) that take place at different locations at subsequent times. Therefore, we are confronted with two interdependent optimal control problems. In order to solve this combined profit-maximising problem, we draw upon the literature of two-stage optimal control problems: notably, Amit (1986), Tomiyama (1985) and Tomiyama and Rossana (1989) provide optimality conditions for two-stage, finite time dynamic optimization problems similar to the one considered here.[8]

We solve this two-stage optimal control problem and derive the optimal travelling–and–harvesting policy, demonstrating the interdependence between the travelling and the harvesting problem, a feature which, to our knowledge, has been left unnoticed and unexplored in the literature. In particular, we show that the need for travelling and thus the presence of the travelling cost requires the agent to weigh the benefit of an early arrival—a long harvesting period and hence a higher yield from harvesting—with its cost: a higher pecuniary travelling cost on the one hand, and also a shorter time for the resource to grow, implying a lower stock upon arrival and hence less beneficial conditions for harvesting. While having a limited amount of time encourages the agent to start harvesting early, the presence of the travelling cost lets the agent postpone the beginning of the harvesting period.[9]

We consider two different possible specifications for the growth process of the resource: exponential growth and logistic growth. Assuming a linear–quadratic travelling cost, we are able to analytically derive the optimal travelling–and–harvesting policies and the associated profits for both types of growth functions, and we show that both policies feature similar characteristics. The fact that we obtain analytic solutions, even in the case of a binding bound on the harvesting capacity is remarkable, as the optimal solution may be composed of different steps, possibly containing time intervals of a singular solution. Finally, we investigate the sensitivity of our results with respect to the presence of bounds on the control of movement and with respect to the rate at which future revenues and costs are discounted.

The rest of the paper is structured as follows: In Section 2 we set up the model. In Section 3 we decompose the travelling–and–harvesting problem into the two sub-problems. We begin our analysis with the harvesting problem in Section 4, and we then proceed with an analysis of the full travelling–and–harvesting problem in Section 5. We conclude in

[8]An extension to an infinite horizon can be found in Makris (2001); applications of this theory to two-stage optimal control problems can be found in Boucekkine et al. (2004), Kim and Park (2008), Saglam (2011), Grass et al. (2012), Caulkins et al. (2013), Krawczyk and Serea (2013), Moser et al. (2014), Aisa et al. (2014), Long et al. (2017), and Seidl et al. (2018).

[9]Beyond Robinson et al. (2008), there are other papers that also acknowledge that travelling time is forgone time for harvesting, *e. g.* Lopez-Feldman and Wilen (2008); Robinson (2016), but these authors either disregard the effect that the time spent on travelling is time for the resource to grow, or focus on the steady state, while we characterise (in a finite time model) the optimal extraction path.

Section 6. The robustness of our results are explored in Appendix A; longer proofs are relegated to Appendix B.

2. The Model

We consider a renewable natural resource situated at some fixed location. The agent who is able to harvest the resource only at its current position, is required to travel to get access to the resource. The agent's problem is thus a combined travelling–and–harvesting problem, where the speed of travelling, and hence the arrival time, and the harvesting rate have to be determined jointly in order to maximise the total profit, composed of the revenue from harvesting net of harvesting and travelling costs.

We consider a finite time horizon T with a planning period $\mathscr{T} \equiv [0, T]$.[10] During this planning period, the economic agent has the exclusive right to harvest the renewable natural resource, which is situated at the fixed and known location $x_1 > 0$. At time $t \in \mathscr{T}$ the location of the economic agent is $x(t) \in \mathscr{X} \equiv [0, \bar{x}]$, where $x_1 \leq \bar{x}$, with the agent's initial location given by $x(0) = 0$. Since the resource is remotely located, at a distance of x_1 units of length from the agent, the agent is unable to begin with harvesting until arrival at location $x(t) = x_1$.

In order to move from one location to the next, the agent has to adjust the velocity of travelling $v(t) \in \mathscr{V} \subset \mathbb{R}$. Since speed cannot be chosen directly, but is physically controlled by means of acceleration $a(t) \in \mathscr{A} \subset \mathbb{R}$ of the vehicle of movement (or the harvesting machine), we have[11]

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = a(t), \quad \forall t \in \mathscr{T} \quad \text{with} \quad x(0) = v(0) = 0. \tag{1a}$$

There may be (binding) lower and upper bounds on acceleration; in Appendix A, we shall assume that acceleration is bounded so that $a \in \mathscr{A} \equiv [\underline{a}, \bar{a}]$ with $\underline{a} < 0$ and $\bar{a} > 0$.[12]

Since both harvesting and travelling take time and the time horizon is finite, the earlier the agent arrives at location x_1 the more time is left for harvesting. In order to render the problem non-trivial, we subsequently assume that the costs of travelling are not too high, so that an arrival before time T is desirable.[13] Since speed is finite, the

[10]Using a finite-time horizon, we follow the initial work on two-stage optimal control problems (see Amit, 1986; Tomiyama, 1985; Tomiyama and Rossana, 1989).

[11]Specifying acceleration, rather than speed, as the control variable avoids the occurrence of unrealistic, *i. e.* discontinuous speed profiles where the agent may abruptly switch speeds.

[12]The minimum acceleration $\underline{a}$ is necessarily negative to allow for a slowdown of speed, as the agent would otherwise be unable to stop—and start harvesting.

[13]If the agent does not arrive at location x_1 by time T, there will be no time left for harvesting—and hence the agent should not travel in the first place.

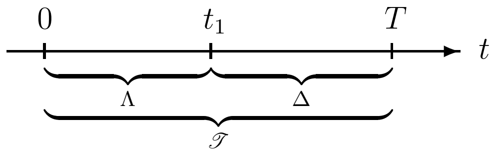

FIGURE 1. Travelling–and–harvesting period

arrival time must be strictly positive. Accordingly, we define

$$t_1 \equiv \min_t \{t \in \mathscr{T} | x(t) = x_1, v(t) = 0\} \tag{1b}$$

as the arrival time of the agent at the location of the resource x_1; that is, $x(t) < x_1$ for all $0 \leq t < t_1$, and $x(t_1) = x_1$ with $v(t_1) = 0$. It follows from our assumptions that t_1 exists and is unique and that both $t_1 = 0$ and $t_1 = T$ can be ruled out; we thus have a unique arrival time $t_1 \in (0, T)$. Consequently, $\Lambda \equiv [0, t_1]$ denotes the agent's travelling period; and $\Delta \equiv (t_1, T]$, the resulting harvesting period.[14] The total time available is then either spent on travelling or on harvesting, so that $\Lambda \cup \Delta = \mathscr{T}$, with $\Lambda, \Delta \neq \emptyset$, represents the *travelling–and–harvesting period*; this is illustrated in Figure 1.

The stock of the renewable resource (*i. e.* the biomass) at time $t \in \mathscr{T}$ is denoted by $s(t) \geq 0$. We assume that the resource is growing at rate $g(s)$ with $g(0) = 0$, so that $s = 0$ represents an equilibrium of the stock dynamics. (Subsequently, we will consider the cases of exponential and logistic growth, both of which satisfy this assumption.) Furthermore, harvesting gradually diminishes the stock. The harvest depends on the abundance of the resource, *i. e.* on the stock s, and on the harvesting effort $h \in \mathscr{H} \equiv [0, \bar{h}] \subset \mathbb{R}_+$. Suppose that the effort is less productive when the stock is lower, with $H = 0$ if $s = 0$, and that a given stock yields less harvest when the effort is lower. To capture this idea, we follow the familiar Schaefer model (see Schaefer, 1954) and specify the revenue from harvesting as a bilinear function of effort and the stock: $H(t) = qs(t)h(t)$, where q is the catchability coefficient, defined as the fraction of the stock harvested per unit of effort. For convenience, we normalise the units of effort to set $q = 1$ such that $H(t) = h(t)s(t)$ provided that the agent's location is x_1, and $H(t) = 0$ otherwise. Thus, the resulting growth of the stock is governed by the differential equation

$$\dot{s}(t) = g(s(t)) - h(t)s(t)\mathbf{1}_\Delta(t), \quad \forall t \in \mathscr{T} \quad \text{with} \quad s(0) = s_0. \tag{1c}$$

where the indicator function $\mathbf{1}_\Delta(t)$ accounts for the fact that harvesting can only be effective if the agent's location at time t equals x_1, *i. e.* if $x(t) = x_1$.

Travelling and harvesting are both costly. We assume that the harvesting cost $C(H)$ is increasing and convex, *i. e.* $C' > 0$ and $C'' \geq 0$ for all $H \in \mathbb{R}_+$, with $C(0) = 0$. Also,

[14] Since the arrival time is unique, the travelling period and the harvesting period are both convex. That is, once the agent has reached location x_1, they will never start travelling again, and thus the agent completes the planning period at x_1, *i. e.* $x(t) = x_1, \forall t \in [t_1, T]$.

travelling is associated with some cost, which generically depends on both speed and acceleration: $K : \mathscr{V} \times \mathscr{A} \to \mathbb{R} : (v, a) \mapsto K(v, a)$. We assume that pausing is costless, $K(0, 0) = 0$, that the travelling cost increases with both speed and acceleration, and that acceleration is more costly the higher the speed, *i. e.* the partial derivatives of K satisfy $K_v \geq 0, K_a \geq 0$ and $K_{va} \geq 0$.

Let $\rho \geq 0$ denote the discount rate of the agent, and let p be the (constant) price of one unit of the harvested resource. The problem for the agent is then to maximise the discounted profit flow consisting of instantaneous revenue net of the harvesting cost and net of the travelling cost for the planning period $\mathscr{T}$. Presupposing that the agent reasonably chooses $h(t) = 0, \forall t \in \Lambda$,[15] we obtain the *travelling cost*

$$J_1(a, t_1) \equiv \int_0^{t_1} e^{-\rho t} K(v(t), a(t))\, \mathrm{d}t$$

and the *profit from harvesting*

$$J_2(h, t_1) \equiv \int_{t_1}^{T} e^{-\rho t} \left(ph(t)s(t) - C(h(t)s(t))\right)\, \mathrm{d}t,$$

where v and x, and hence the arrival time t_1 depend on the acceleration path $\{a\}_{t \in \Lambda}$. Putting the pieces together, the agent's optimisation problem then reads as

$$\max_{\{a,h,t_1\}} \quad J(a, h, t_1) \equiv -J_1(a, t_1) + J_2(h, t_1) \tag{1d}$$

subject to the dynamics of movement (1a), the arrival time (1b), the stock dynamics (1c), and their associated constraints $v(t) \in \mathscr{V}(t)$, $a(t) \in \mathscr{A}(t)$, $h(t) \in \mathscr{H}(t)$, $\forall t \in \mathscr{T}$, and the free terminal condition $s(T) \geq 0$.[16]

3. Decomposition of the problem

In order to solve problem (1), we decompose the intertemporal optimal travelling–and–harvesting problem into two subsequent problems: a travelling and a harvesting sub-problem. In the *travelling problem* we choose an acceleration path $\{a(t)\}_{t \in \Lambda}$, and the arrival time t_1, so as to move from location 0 to location x_1 at minimal cost:

$$\min_{\{a \in \mathscr{A}, t_1 \in \mathscr{T}\}} J_1(a, t_1) \quad \text{s. t.} \quad (1a), (1b) \tag{2a}$$

Since the travelling time t_1 can be chosen subject to the constraint $x(t_1) = x_1$, we face a free-terminal-time problem with a fixed endpoint constraint.

During the travelling period Λ, the resource grows unimpaired until the agent arrives at the location x_1 at time t_1, when the harvesting period begins. As a consequence, the

[15] We may allow the agent to choose $h(t) > 0$ for times $t < t_1$, but since this harvesting activity is sure to yield no return at any time $t \in \Lambda$, the choice of $h(t) > 0$ is a futile action in this case.

[16] The constraints $x(t) = x_1$ and $v(t) = 0, \forall t \in \Delta$ are already implied by (1b) and (1c).

stock of the resource at the time of arrival, $s(t_1)$, represents the solution of the (interim) growth process $\dot{s}(t) = g(s(t))$ with $s(0) = s_0$ for all $t \in \Lambda$. In this way, the travelling decision determines the initial value of the stock process for the harvesting problem, $s_1 = s(t_1)$. Given those parameters, the resulting subsequent *harvesting problem* becomes choosing a path of the harvesting activity (effort) $\{h(t)\}_{t\in\Delta}$ that maximises J_2:

$$\max_{\{h\in\mathscr{H}\}} J_2(h, t_1) \quad \text{s. t.} \quad \text{(1c)},\ s(T) \text{ free.} \tag{2b}$$

The fact that the travelling time of the agent also represents the growth time of the resource is the crucial link between the travelling problem (2a) and the harvesting problem (2b). As a consequence, the agent has to take into account that a longer travelling time will reduce the time left for harvesting, and thus *ceteris paribus* the resulting yield. In contrast, a lower speed of travelling makes travelling less expensive and gives the resource more time to grow, thus providing the opportunity for a more abundant harvest at later times. The *travelling–and–harvesting problem* (1) takes into account these interdependencies between sub-problems (2a) and (2b).

We derive necessary conditions for an optimal control pair (a^*, h^*, t_1^*) by decomposing the original problem (1) into two standard problems. We first consider the harvesting problem of the second stage (2b), and then the travelling problem of the first stage (2a), acknowledging that the solution of the second stage depends on the decision in the first stage. Formally we proceed as follows: Assuming the existence of the optimal switching time t_1 in the interior of the time interval $\mathscr{T}$, we solve the second–stage problem and calculate the maximised objective function J_2^* as a function of the initial state s_1 and the switching time t_1. Then, we derive the optimal control a^* and the associated optimal switching time t_1 by solving the travelling problem of the first stage.[17]

Second stage. Given the control time interval $[t_1, T]$ and the initial condition $s(t_1) = s_1$, we solve problem (2b) for an admissible optimal control h^*. This problem is of a standard form and can be solved using the well-known Pontryagin maximum principle (see, for example, Kamien and Schwartz, 1991.) Using the solution of the second-stage problem, h^*, λ_2^* and s^*, which depends on the starting values s_1 and t_1, we calculate $J_2^*(s_1, t_1) \equiv J_2(h^*(s_1, t_1), t_1)$. Then, with the help of J_2^*, the original problem (1) reduces to the *first–stage problem*:

[17]Tomiyama and Rossana (1989) and Grass et al. (2008, sec. 8.1.1) generalise the results of Tomiyama (1985) and Amit (1986) for a finite and an infinite time horizon, respectively, when the switching point appears as an argument of the integrands in each integral of the objective function.

First stage. Given the constraints (1a) and (1b), we look for an admissible optimal control a^* on $[0, t_1^*]$ and the associated optimal arrival time $t_1^* \in (0, T)$ such that

$$\max_{\{a \in \mathscr{A}, t_1 \in \mathscr{T}\}} V(a, t_1) \equiv -J_1(a, t_1) + J_2^*(s(t_1), t_1), \tag{3}$$

where the optimal arrival time t_1^*, satisfying (1b), is determined by the acceleration path $\{a^*\}_{t \in \Lambda}$, through $\int_0^{t_1} v(t) \mathrm{d}t = x(t_1) - x(0) = x_1$. Since $t_1^* \in (0, T)$ by assumption, this problem reduces to a standard problem with 'scrap' value J_2^*, free terminal time t_1 and free end point $s(t_1)$. (See, for example, Léonard and Long, 1992, sec. 7.2 and 7.6.) The optimality condition for this type of two-stage dynamic optimization problem can be found in Tomiyama (1985) and Amit (1986) who provide the necessary conditions we employ here.

4. Second Stage: Harvesting

We solve the harvesting problem in this section, and then we solve the problem of the second stage in Section 5. We consider two standard specifications for the growth process of the resource: exponential growth in sub-section 4.1 and logistic growth in sub-section 4.3. For both processes we derive the value function of the optimal harvesting policy.[18]

4.1. Exponential growth. Suppose that the stock of the renewable resource, when left unimpaired, increases at a constant rate: $g(s(t)) = s(t)$ for all $t \in \Delta \equiv [t_1, T]$. Since the stock is reduced by the catch $H(t) \equiv s(t)h(t)$, the stock evolves according to the differential equation

$$\dot{s}(t) = s(t) - h(t)s(t), \quad s(t_1) = s_1, \quad \forall\, t \in \Delta,\, h(t) \in \mathscr{H}. \tag{4a}$$

To complete our definition of the profit function, we follow the specification of the effort cost function chosen by, for example, Puchkova et al. (2014) and Moberg et al. (2015) and assume that harvesting costs are linear in total catch, $C(H(t)) = cH(t) = c\, h(t)s(t)$, with $0 \leq c < p$.[19] Then, instantaneous profit amounts to $(p - c)h(t)s(t)$. Finally, writing

[18]Similar models can be found in Conrad and Clark (1987); Hocking (1991); Clark (2010).

[19]The Schaefer model is commonly used in the literature, and many authors add either linear or quadratic effort cost. For example, Clark (2010, Sec. 1.4), Puchkova et al. (2014) and Moberg et al. (2015) assume linear cost yielding an instantaneous profit equal to $pqs(t)h(t) - ch(t)$; conversely, He et al. (1994), Leung (1995), Cañada et al. (2001), Montero (2001), Fister and Lenhart (2004, 2006) and Chang and Wei (2012) presume a quadratic effort cost function, and Ding and Lenhart (2009) presume a linear-quadratic effort cost function. One exception to the prevalence of linear and quadratic cost functions is Liski et al. (2001), who suppose a concave-convex harvest cost. An alternative specification of the objective function is to disregard effort cost altogether and to maximise the sustainable yield; this approach is followed by, for example, Fan and Wang (1998), Neubert (2003), Bai and Wang (2005) and Kelly et al. (2016).—All of these authors assume a fixed price of the (harvested) resource.

$M \equiv p - c$ for the per–unit profit (mark-up), the objective function of (2b) becomes

$$\max_{\{h \in \mathscr{H}\}} J_2(h, t_1) = \int_{t_1}^{T} Mh(t)s(t)\,\mathrm{d}t \quad \text{s.t. (4a), } s(T) \text{ free.}$$

We abstract from discounting for the moment and set $\rho = 0$, as does the majority of references provided in fn. 19. We show in Appendix A how our results are affected by the presence of a positive discount rate.

The Hamiltonian of this problem is given by

$$\mathcal{H} = Mh(t)s(t) + \pi(t)s(t)\left(1 - h(t)\right).$$

Since $\mathcal{H}$ is linear in the control h, we expect that the optimal solution follows a *most rapid approach path*, with h jumping between the lower and the upper, 0 and $\bar{h}$. We now show that such a policy is in fact optimal. Using the Hamiltonian, the maximum principle yields

$$0 = (M - \pi(t))s(t), \qquad \forall\, t \in \Delta, \tag{4b}$$

$$\dot{\pi}(t) = h(t)\pi(t) - Mh(t) - \pi(t), \quad \forall\, t \in \Delta, \tag{4c}$$

along with eq. (4a) and the transversality condition $\pi(T) = 0$.[20] It follows from eq. (4b) that the optimal strategy depends on whether π is less or greater than one. The maximum of $\mathcal{H}$ is thus achieved by

$$h(t) = \begin{cases} 0 & \text{if } \pi(t) > M \\ \bar{h} & \text{if } \pi(t) < M. \end{cases} \tag{5a}$$

Using $\pi(T) = 0$ together with $h(t) = \bar{h}$ for $\pi(t) < M$, implies that we cannot end the period Δ with $h = 0$, *i. e.* we must have $h(T) = \bar{h}$. Moreover, the solution of eq. (4c) must satisfy

$$\pi(t) = \begin{cases} A_0 e^{-t} & \text{if } h(t) = 0 \\ M\dfrac{\bar{h}}{\bar{h} - 1} + A_1 e^{t(\bar{h}-1)} & \text{if } h(t) = \bar{h}. \end{cases}$$

Neither solution achieves the critical value $\pi = M$ more than once. Consequently, there is a unique switching point τ,[21] implying that we either have (i) $h(t) = \bar{h}$ for all $t \in \Delta$, or

[20] The use of the transversality condition $\pi(T) = 0$ requires that $s(T)$ is free. Of course, $s(T)$ is required to be nonnegative, so that, in principle, $\pi(T)$ may be positive if $s^*(T) = 0$. Hence, we proceed by assuming, for the moment, that $s(T)$ is free, and afterwards check that $s^*(T) > 0$, justifying the transversality condition $\pi(T) = 0$. For details see, for example, Seierstad and Sydsæter (1987, Theorem 2.1 and 2.2) or Grass et al. (2008, Theorem 3.4 and 3.14).

[21] Alternatively, this observation follows from eq. (4c), which implies that evaluated at a switching point τ, we have $\dot{\pi}(\tau) = -M$ since $\pi(\tau) = M$ by definition.

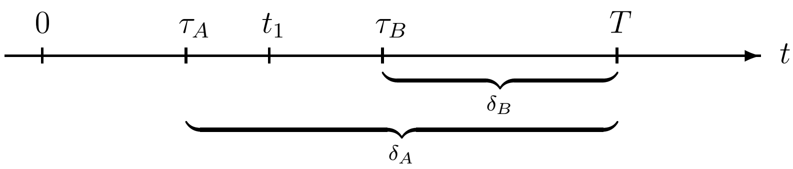

FIGURE 2. Cases A and B

(ii) $h(t) = 0$ for all $t_1 \leq t < \tau$ and $h(t) = \bar{h}$ for all $\tau \leq t \leq T$. Then, along any path with $h = \bar{h}$, the costate variable is given by

$$\pi(t) = M \frac{\bar{h}}{\bar{h}-1} \left(1 - e^{(1-\bar{h})(T-t)}\right) \tag{5b}$$

where we used the transversality condition $\pi(T) = 0$ to determine the constant $A_1 = M\bar{h}e^{T(1-\bar{h})}/(1-\bar{h})$. Moreover, the switching time τ is defined by $\pi(\tau) = M$ so that we obtain from eq. (5b)

$$\tau = T - \delta, \quad \text{with} \quad \delta \equiv \frac{\log\left(\bar{h}\right)}{\bar{h}-1}. \tag{5c}$$

Since δ is a positive, decreasing and convex function for all values of $\bar{h} \neq 1$, we define $\delta = 1$ for $\bar{h} = 1$ so as to make δ a continuous function of $\bar{h}$.[22] Consequently, the larger $\bar{h}$, the longer the agent can wait and let the resource grow unimpaired, allowing for more intensive harvesting later. Depending on the sign of $\tau - t_1$, either of two cases may occur:

4.1.1. *Case A: $T < \delta + t_1$.* In this case the maximal harvesting intensity $\bar{h}$ is relatively low, requiring a rather long period of harvesting: $T - t_1 < \delta \Leftrightarrow \tau < t_1$. There is thus no time for "waiting" by setting $h(t) = 0$ after arrival at time t_1, and hence no policy switch in Δ.

Proposition 1. *Let $T < \delta + t_1$ and $\bar{h} \neq 1$. Then the optimal harvesting policy is given by*

$$h(t) = \bar{h}, \quad s(t) = s_1 e^{(1-\bar{h})(t-t_1)}, \quad \pi(t) = M\frac{\bar{h}}{\bar{h}-1}\left(1 - e^{(1-\bar{h})(T-t)}\right), \tag{6a}$$

for all $t \in \Delta$, and the resulting maximised profit amounts to

$$J^*_{2A}(s_1, t_1) \equiv s_1 M \frac{\bar{h}}{\bar{h}-1}\left(1 - e^{(1-\bar{h})(T-t_1)}\right). \tag{6b}$$

Proof. The result follows from the preceding analysis. ☐

[22] To see that $\delta = 1$ for $\bar{h} = 1$, apply l'Hôpital's rule.

4.1.2. *Case B: $T > \delta + t_1$.* In this case the maximal harvesting intensity $\bar{h}$ is relatively high, so that the agent may afford not to begin with harvesting immediately at time t_1 but at some point in time: $T - t_1 > \delta \Leftrightarrow \tau > t_1$. Hence, there is time for "waiting" and the agent begins with $h = 0$ and then, at time τ, switches to $h = \bar{h}$. During the period $[t_1, \tau)$ the stock is left unimpaired and is thus given by $s(t) = s_1 e^{t-t_1}$, so that at time τ the stock amounts to $s(\tau) = s_1 e^{\tau - t_1}$, which is the starting value for the period $[\tau, T]$; for times $t \in [\tau, T]$ the stock equals

$$s(t) = A_2 e^{(1-\bar{h})t} = s(\tau) e^{\left(1-\bar{h}\right)(t-\tau)} = s_1 e^{\bar{h}(\tau - t) + t - t_1}.$$

Proposition 2. *Let $T > \delta + t_1$ and $\bar{h} \neq 1$. Then the optimal harvesting policy is given by*

$$h(t) = \begin{cases} 0 & \text{for } t_1 \leq t < \tau \\ \bar{h} & \text{for } \tau \leq t \leq T \end{cases} \tag{7a}$$

$$s(t) = \begin{cases} s_1 e^{t-t_1} & \text{for } t_1 \leq t < \tau \\ s_1 e^{\bar{h}(\tau-t)+t-t_1} & \text{for } \tau \leq t \leq T \end{cases} \tag{7b}$$

$$\pi(t) = \begin{cases} e^{\tau - t} & \text{for } t_1 \leq t < \tau \\ M \frac{\bar{h}}{\bar{h}-1}\left(1 - e^{\left(1-\bar{h}\right)(T-t)}\right) & \text{for } \tau \leq t \leq T, \end{cases} \tag{7c}$$

for all $t \in \Delta$, and the maximised profit amounts to

$$J^*_{2B}(s_1, t_1) \equiv M \frac{\bar{h}}{\bar{h}-1} s_1 e^{\tau - t_1}\left(1 - e^{(1-\bar{h})(T-\tau)}\right) = M s_1 \, \bar{h}^{1/(1-\bar{h})} \, e^{T-t_1}. \tag{7d}$$

Proof. The result follows from the preceding analysis. □

4.1.3. *The case of $\bar{h} = 1$.* Finally, the optimal policy for the case $\bar{h} = 1$ is obtained by taking the limits of Case A and Case B:

Remark 1. If $\bar{h} = 1$, the optimal profit amounts to

$$J^*_2|_{\bar{h}=1}(s_1, t_1) = \begin{cases} s_1 M (T - t_1) & \text{if } T \leq \delta + t_1 \\ s_1 M e^{T - t_1 - 1} & \text{if } T > \delta + t_1. \end{cases}$$

4.2. Discussion. In the optimal solution, the length of the harvesting period equals $\delta = T - \tau = \log(\bar{h})/(\bar{h} - 1)$. If there is plenty of time in the sense that $T > \delta + t_1$, there will be no harvesting during the initial period $[t_1, \tau)$, while harvesting will take place at the maximum rate $\bar{h}$ during the final period $[\tau, T]$. If, however, there is not enough time available, that is if $T \leq \delta + t_1$, the agent harvests all the time at the maximum rate $\bar{h}$. Whether the stock increases or decreases during harvesting, depends on whether the harvesting capacity $\bar{h}$ exceeds or falls short of the growth rate of the stock, which is

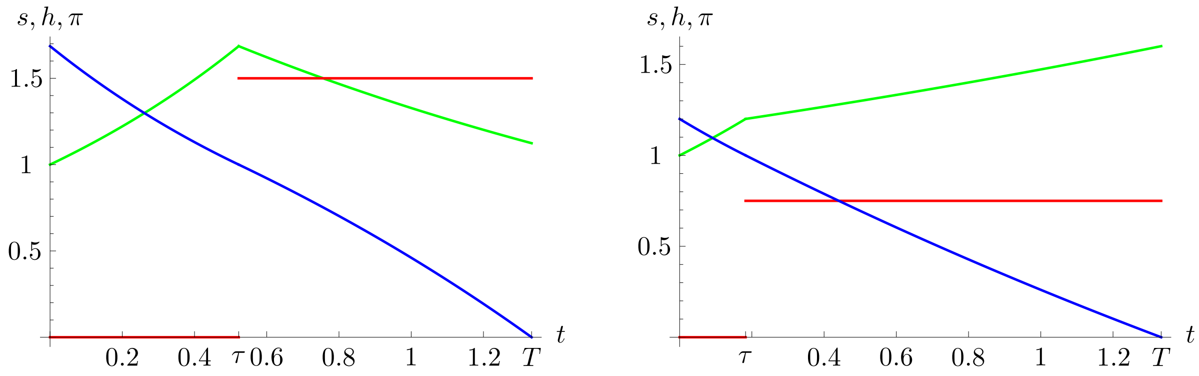

FIGURE 3. Optimal harvesting in Case B, $T > \delta + t_1$, with $t_1 = 0$: $\bar{h} > 1$ (left) and $\bar{h} < 1$ (right), with the stock in green, the costate in blue and the control in red; both for $M = 1$.

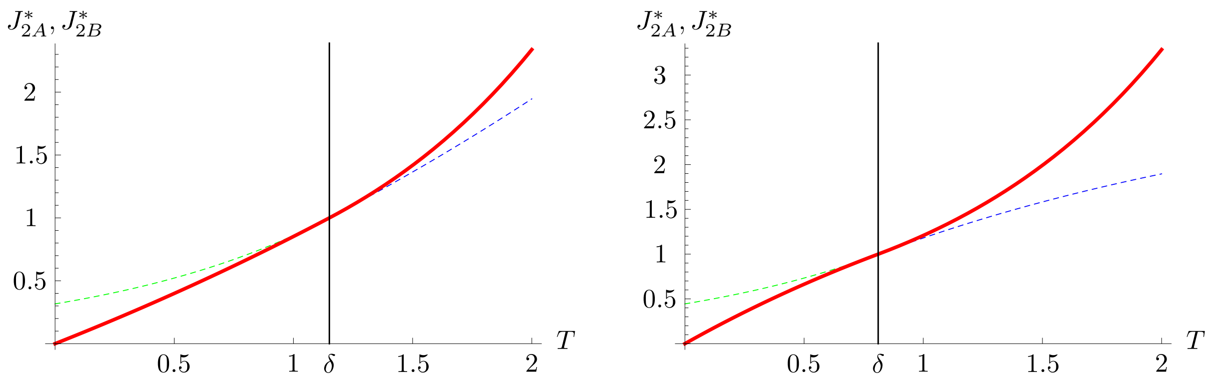

FIGURE 4. Maximised profit function for $\bar{h} = 3/4 < 1$, *i. e.* $\delta = 4\log\left(\frac{4}{3}\right) = 1.15073$ (left); and for $\bar{h} = 3/2 > 1$, *i. e.* $\delta = 2\log\left(\frac{3}{2}\right) = 0.81093$ (right); both for $M = 1$.

assumed to be equal to 1 here. The situation when $\bar{h} < 1$ is depicted in the left diagram of Figure 3, and when $\bar{h} > 1$, in the right diagram (both for $t_1 = 0$).

Remarkably, the optimal length of the harvesting period, δ, depends on $\bar{h}$ but is independent of T; the maximised profit though (in both Case A and B) depends on T. While J^*_{2B} is increasing and convex in T, J^*_{2A} is convex only if $\bar{h} < 1$, and it is concave if $\bar{h} > 1$. Moreover, for any given values of t_1 and s_1 we have $J^*_{2B} \geq J^*_{2A}$. This is depicted in Figure 4 for the case $t_1 = 0$. Therein, the vertical line represents the critical time $T = \delta + t_1$ for a given value of $\bar{h}$, and the red curve depicts the (composed) profit function for varying values of T. If time is scarce in the sense that $T - t_1 < \delta$, Case A applies and the blue curve represents the resulting maximised profit (coinciding with the red curve for values $T < \delta$). If there is plenty of time, in the sense that $T > \delta + t_1$, Case B applies and the green curve represents the resulting maximised profit (similarly coinciding with the red curve for values $T > \delta + t_1$).

4.3. Logistic growth. In this section we modify the growth process of the resource and now assume that the stock obeys a logistic growth process:

$$g(s(t)) = 2s(t)\left(1 - \frac{s(t)}{2}\right), \quad \forall t \in \Delta,$$

i. e. we have a standard, strictly concave growth function, with $g(0) = g(2) = 0$, where $s^* = 2$ is the "carrying capacity".[23] With this specification, the net-growth of the stock is governed by the differential equation

$$\dot{s}(t) = g(s(t)) - h(t)s(t) = s(t)\,(2 - s(t) - h(t)) \tag{8a}$$

We assume that the resource is left unimpaired for a sufficiently long time, so that at time t_1 the stock equilibrates at its steady state level $s(t_1) = s^* = 2$. The remaining model is adopted from Section 4.1.

The Hamiltonian of the problem is given by

$$\mathcal{H} = Mh(t)s(t) + \pi(t)s(t)\,(2 - s(t) - h(t))\,,$$

As in the case of exponential growth, $\mathcal{H}$ is linear in the control h. Yet here, as we will see, the optimal solution is not only of the bang-bang type, but may also follow a *singular path* for some time interval if there is plenty of time (or, equivalently, if $\bar{h}$ is sufficiently large). Using the Hamiltonian, the maximum principle yields

$$0 = (M - \pi(t))s(t), \qquad \forall\, t \in \Delta, \tag{8b}$$

$$\dot{\pi}(t) = -Mh(t) - \pi(t)\,(2 - 2s(t) - h(t))\,, \quad \forall\, t \in \Delta, \tag{8c}$$

together with eq. (8a) and the transversality condition $\pi(T) = 0$.

Lemma 1. $\pi(t_1) < M$.

Proof. See Appendix B. □

It follows from Lemma 1 that the optimal policy rule coincides with the rule obtained for exponential growth of the resource (5a): The maximum of the Hamiltonian $\mathcal{H}$ is achieved by

$$h(t) = \begin{cases} 0 & \text{if } \pi(t) > M \\ \bar{h} & \text{if } \pi(t) < M. \end{cases} \tag{9}$$

Since $\pi(t_1) < M$ by Lemma 1, the optimal path begins with $h(t_1) = \bar{h}$. Intuitively, since the initial stock equals its maximum level, $s(t_1) = 2$, there is no reason to begin with $h = 0$. If time is scarce, relative to the harvesting capacity, we continue with $h(t) = \bar{h}$ for all $t \in \mathscr{T}$; while if there is plenty of time, it is optimal to reduce harvesting in the

[23]This specification is used, for example, in Pindyck (1984), Conrad and Clark (1987), Thieme (2003), Da Lara and Doyen (2008), Polasky et al. (2011) and innumerable other applications.

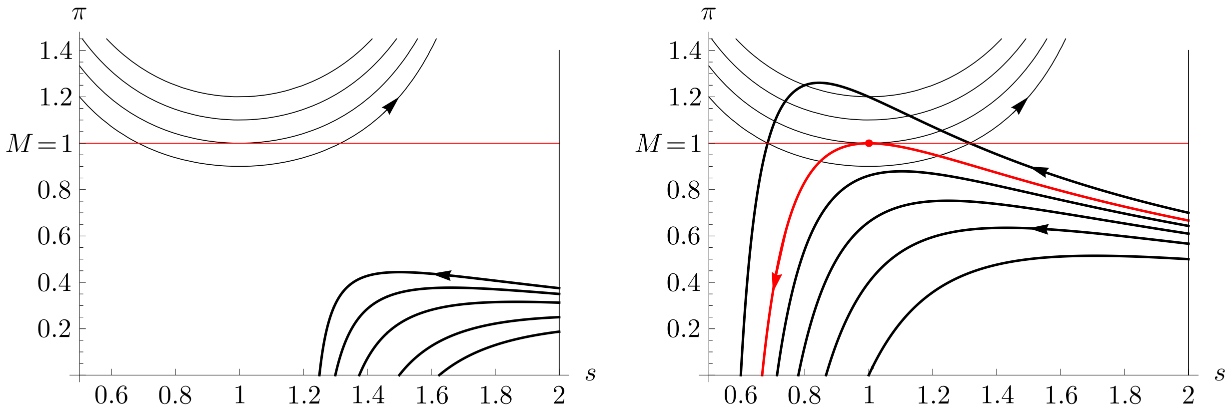

FIGURE 5. Left: Trajectories for $h = \bar{h} = 0.8$ (bold curves) and $h = 0$ (thin curves); right: Trajectories for $h = \bar{h} = 1.5$ (bold curves) and $h = 0$ (thin curves), with the critical trajectory (red); both for $M = 1$.

meantime, because otherwise we would have completed harvesting too early, and the terminal condition $\pi(T) = 0$ will not be met.

Proposition 3. *The optimal harvesting policy is given by*

$$h(t) = \bar{h} \qquad\qquad \text{if} \quad \bar{h} \leq \bar{h}_c \tag{10a}$$

$$h(t) = \begin{cases} \bar{h} & t_1 \leq t < t_2, \\ 1 & t_2 \leq t < t_3, \\ \bar{h} & t_3 \leq t < T. \end{cases} \quad \text{if} \quad \bar{h} > \bar{h}_c, \tag{10b}$$

with some critical harvesting capacity $\bar{h}_c > 1$ (depending on T).

Proof. See Appendix B. □

Figure 5 displays two types of trajectories: one for a small harvesting capacity (left diagram) and one for a high harvesting capacity (right diagram). The trajectories starting from $s(t_1) = 2$ reach the horizontal axis at time T, as the transversality condition requires $\pi(T) = 0$. If $\bar{h}$ is small, the trajectory does not reach the $\pi = M(= 1)$ line (see Figure 5, left), while if $\bar{h}$ is sufficiently large, it does. In fact, the proof of Proposition 3 shows that there exists a critical trajectory that touches the $\pi = M(= 1)$ line, and that this trajectory must feature $\bar{h} > 1$. For this reason, the critical harvesting capacity must exceed unity, *i. e.* $\bar{h}_c > 1$. Moreover, the critical harvesting capacity $\bar{h}_c$ depends inversely on the time horizon T.

Lemma 2. *Let $\psi : (1, 2] \to \mathbb{R}_+$ be defined by*

$$\bar{h} \mapsto \psi(\bar{h}) \equiv t_1 + \frac{1}{2 - \bar{h}} \log\left(\frac{\bar{h}}{2(\bar{h} - 1)^2}\right) \tag{11}$$

with $\Psi > 0$, $\Psi' < 0$ *and* $\Psi'' > 0$. *Then, given time* T, *the* critical harvesting capacity $\bar{h}_c$ *is defined as the solution of* $T = \psi(\bar{h})$, i.e. $\bar{h}_c \equiv \psi^{-1}(T)$. *Equivalently, given some harvesting capacity* $\bar{h}$, *the* critical length of the harvesting period *is defined by* $T_c \equiv \psi(\bar{h})$.

Proof. See Appendix B. □

The intuition for the optimal strategy characterised in Proposition 3 and the critical harvesting period given in Lemma 2 is as follows. In the case $T > T_c$, there is too much time for harvesting, implying that if the agent followed the critical path (the red path in the right diagram of Figure 5), they would have reached the $\pi = 0$ line too early. Thus, one might consider following a trajectory lying above the critical one, reaching the $\pi = M$ line at some value $s > 1$. But then one has to switch to $h = 0$ following an upward-sloping trajectory (a thin path in the right diagram of Figure 5), implying that both the stock and the costate variable increase; and, satisfying the terminal condition $\pi(T) = 0$ becomes impossible. For that reason the optimal policy is as follows: pursue the critical path up to $(s, \pi) = (1, M)$, which is reached at time t_2; then, upon arrival at $(s, \pi) = (1, M)$, reduce harvesting to $h = 1$, which, in view of eqs (8a) and (8c), renders both s and π to be constant, for 1 is the natural growth rate of the resource; finally, to complete the optimal path, resume maximal harvesting so as to arrive at $\pi = 0$ at time T.

4.3.1. *Case A: either* $\bar{h} < 1$ *or* $1 < \bar{h} < 2$ *and* $T \le T_c$. In this case, the maximal harvesting effort is relatively low, $\bar{h} < \bar{h}_c = \psi^{-1}(T)$, so that $h(t) = \bar{h}$ can be maintained throughout. Then, the optimal harvesting strategy is given by:

Proposition 4. *Let either* $\bar{h} < 1$ *or* $1 < \bar{h} < 2$ *and* $T \le T_c$. *Then the optimal harvesting policy is given by*

$$h(t) = \bar{h}, \quad s(t) = \frac{2\left(\bar{h} - 2\right)}{\bar{h}e^{\left(\bar{h}-2\right)(t-t_1)} - 2}, \quad \pi(t) = M\frac{\bar{h}(s(T) - s(t))}{2s(t) - s(t)^2 - \bar{h}s(t)}, \tag{12a}$$

for all $t \in \Delta$, *and the resulting maximised profit amounts to*

$$J^*_{2A}(t_1) = \bar{h}M\int_{t_1}^{T} s(t)\,\mathrm{d}t = \bar{h}M\log\left(\frac{2e^{(2-\bar{h})(T-t_1)} - \bar{h}}{2 - \bar{h}}\right). \tag{12b}$$

Proof. We know from the proof of Proposition 3 that for all sub-critical cases $T < T_c$ (or $\bar{h} < \bar{h}_c$) defined in eq. (11), we have $h(t) = \bar{h}$ for all $t \in \Delta$. Substituting this, jointly with initial condition $s(t_1) = 2$ and the terminal condition $\pi(T) = 0$, into eqs (8a)–(8c) we obtain (12a). □

Remark 2. For the limiting case when $\bar{h} \to 1$, the resulting profit amounts to $J^*_{2A}|_{\bar{h}=1}(t_1) = M\log\left(2e^{T-t_1} - 1\right)$.

Remark 3. In the critical case, *i. e.* when $T = T_c$, the optimal profit amounts to

$$J_{2A}^c(t_1) = 2\bar{h}M \log\left(\frac{\bar{h}}{\bar{h}-1}\right). \tag{13}$$

4.3.2. *Case B:* $1 < \bar{h} < 2$ *and* $T > T_c$. In this case, the time available for harvesting $T - t_1$ is too long such that, given the maximal harvesting capacity $\bar{h}$, it is not optimal to harvest at the maximal rate all the time, as this would imply that $\pi = 0$ is reached before time T. Thus, harvesting cannot be maintained at rate $\bar{h}$ throughout, but must be reduced during some interval.

Proposition 5. *Let* $1 < \bar{h} < 2$ *and* $T > T_c$. *Then the optimal harvesting policy is given by*

$$h(t) = \begin{cases} \bar{h} & t_1 \leq t < t_2, \\ 1 & t_2 \leq t < t_3, \\ \bar{h} & t_3 \leq t < T, \end{cases} \tag{14a}$$

with switching times

$$t_2 = t_1 + \frac{\log\left(\frac{\bar{h}}{2(\bar{h}-1)}\right)}{2-\bar{h}} \quad \textit{and} \quad t_3 = T - \frac{\log\left(\frac{1}{\bar{h}-1}\right)}{2-\bar{h}}.$$

The resulting profit is given by

$$J_{2B}^*(t_1) = M\left[T - t_1 + 2\bar{h}\log\left(\frac{\bar{h}}{\bar{h}-1}\right) - \frac{1}{2-\bar{h}}\log\left(\frac{\bar{h}}{2(\bar{h}-1)^2}\right)\right]. \tag{14b}$$

Proof. See Appendix B. □

In the limiting case where $T = T_c$, we have $t_2 = t_3$ and the central interval vanishes. More generally, since $\partial t_2/\partial\bar{h} < 0$ and $\partial t_3/\partial\bar{h} > 0$, the central interval increases with $\bar{h}$. The reason for this is that a higher harvesting capacity allows the agent to harvest more intensively in the beginning and at the end of the harvesting period, so that harvesting must be reduced in the central time interval. Yet, since $h(t) = 1$ is fixed for all $t \in [t_2, t_3)$, the only way to accomplish a lower catch in the central time interval is to extend this interval.

Remark 4. For the limiting case when $\bar{h} \to 1$, the resulting profit equals $J_{2B}^*|_{\bar{h}=1} = M[T - t_1 + \log(2)]$; while for the case $\bar{h} \to 2$, the profit amounts to $J_{2B}^*|_{\bar{h}=2} = M\left[T - t_1 - \frac{3}{2} + \log(16)\right]$. Finally, when the harvesting capacity becomes unbounded, *i. e.* $\bar{h} \to \infty$, we obtain $J_{2B}^* = M[T - t_1 + 2]$. Hence, we have $J_{2B}^*|_{\bar{h}=1} < J_{2B}^*|_{\bar{h}=2} < J_{2B}^*|_{\bar{h}=\infty}$, as expected.

As Remark 4 suggests, the maximised profit function J_2^* is increasing in the capacity $\bar{h}$; this is depicted in Figure 6 for $M = 1$, $T = 2$, 5 and 20. Therein, the vertical line

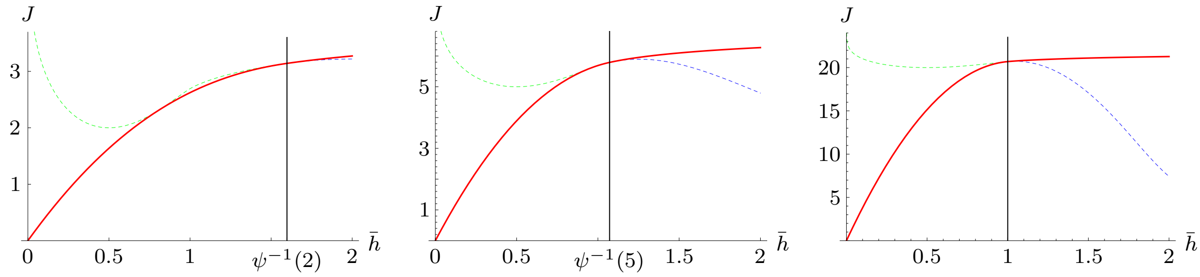

FIGURE 6. Profit in Case A and B for $T = 2$ (left), 5 (middle), 20 (right) with varying values of $\bar{h}$.

represents the critical capacity $\bar{h}_c = \psi^{-1}(T)$. For values of $\bar{h} < \psi^{-1}(T)$, Case A applies; for values of $\bar{h} > \psi^{-1}(T)$, Case B. The critical values $\bar{h}_c = \psi^{-1}(T)$ can be gathered from eq. (11).

5. First Stage: Optimal travelling–and–harvesting policy

Having solved the harvesting problem, we now go back in time and solve the travelling problem. We begin our analysis with the simple, hypothetical case of a fixed travelling period in sub-section 5.1, and then continue with acknowledging the subsequent harvesting period and endogenising the arrival time t_1 in sub-section 5.2. We proceed in this successive manner, for this allows us to spotlight the differences between the solution of the isolated travelling problem (2a) and the solution of the travelling–and–harvesting problem (3).

5.1. Fixed travelling period. Assume that the cost of travelling depends linearly on speed v and quadratically on acceleration a:[24]

$$K(v, a) = cv + a^2. \tag{15}$$

Again assuming $\rho = 0$, the resulting aggregated travelling cost amounts to

$$\int_0^{t_1} \left(cv(t) + a(t)^2\right) \, \mathrm{d}t. \tag{16}$$

(In Appendix A we explore the effects of a positive discount rate.) Acknowledging the constraints $\dot{x}(t) = v(t)$, $\dot{v}(t) = a(t)$, and $\dot{s}(t) = g(s(t))$, we obtain the Hamiltonian

$$\mathcal{H}_1 = -cv(t) - a(t)^2 + \pi_2(t)a(t) + \pi_1(t)v(t).$$

[24]A linear-quadratic dependence of fuel consumption on the kinematic variables velocity and acceleration (along with the characteristics of the road, motor data etc.) is typically used and has been empirically tested in transport economics. See, for example, Ahn et al. (2002); Bifulco et al. (2015); Wörz and Bernhardt (2017) and the references therein.

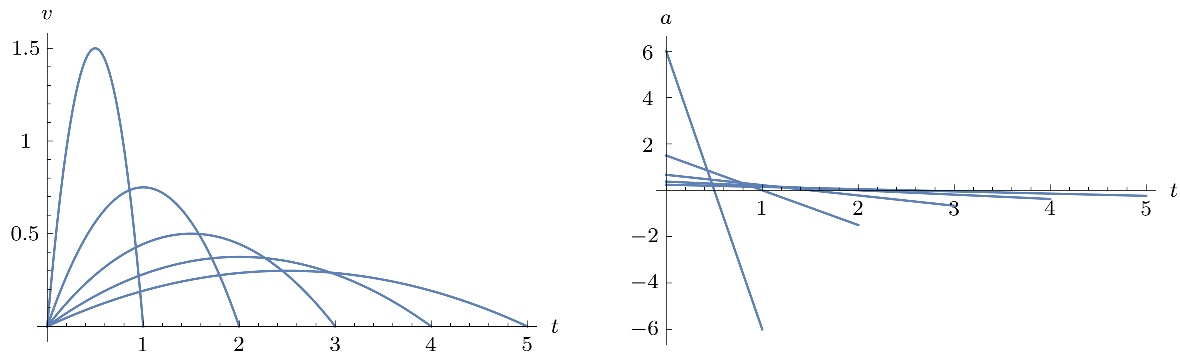

FIGURE 7. Speed (v) and acceleration (a) for $t_1 = 1, \dots, 5$ ($c = 1/10, x_1 = 1$).

For ease of tractability, we assume that there are no bounds on the control a—yet, we will drop this assumption in Appendix A. The familiar maximum principle then yields

$$x(t) = \frac{t^2}{12}\left(3K_1 - K_2 t + ct\right), \qquad v(t) = \frac{t}{4}\left(2K_1 - K_2 t + ct\right),$$
$$\pi_1(t) = K_2, \qquad \pi_2(t) = K_1 + t\left(c - K_2\right),$$

with K_1 and K_2 constants. Together with the boundary conditions $x(0) = v(0) = v(t_1) = 0$ and $x(t_1) = x_1$, we obtain:

Proposition 6. *Given arrival time t_1, the optimal travelling policy is given by*

$$x(t) = \frac{t^2\left(3t_1 - 2t\right)x_1}{t_1^3}, \qquad v(t) = \frac{6t\left(t_1 - t\right)x_1}{t_1^3}, \qquad a(t) = \frac{6\left(t_1 - 2t\right)x_1}{t_1^3},$$
$$\pi_1(t) = c + \frac{24x_1}{t_1^3}, \qquad \pi_2(t) = \frac{12\left(t_1 - 2t\right)x_1}{t_1^3},$$

and the minimised objective function equals

$$J_1^*(t_1) = \int_0^{t_1} \left(a(t)^2 + cv(t)\right)\, \mathrm{d}t = cx_1 + \frac{12x_1^2}{t_1^3}. \tag{17}$$

Proof. The result follows from the preceding analysis. □

Since J_1^* enters the objective function negatively, the value of the maximised Hamiltonian equals

$$\mathcal{H}_1^*(t_1) \equiv \mathcal{H}_1(s(t_1), a(t_1), \pi(t_1), t_1) = -\frac{\mathrm{d}J_1^*(t_1)}{\mathrm{d}t_1} = \frac{36x_1^2}{t_1^4}. \tag{18}$$

The acceleration of the vehicle and its resulting speed are depicted in Figure 7 for varying arrival times t_1.

5.2. Optimal travelling period. In sub-section 5.1 we assumed that t_1 is fixed. However, the agent may choose the length of the travelling period and, concordantly, the beginning of the harvesting period. In order to determine the optimal policy for the travelling–and–harvesting problem, two different effects must be taken into account, and the associated conditions have to be added to those of the pure travelling decision. First, the growth process of the resource during the travelling period must be acknowledged, and the corresponding necessary optimality condition needs to be added to the canonical system:

$$\dot{s}(t) = g(s(t)) = \begin{cases} 2s(t) - s^2(t) & \text{logistic growth} \\ s(t) & \text{exponential growth,} \end{cases} \tag{19a}$$

$$\dot{\pi}(t) = -\frac{\partial \mathcal{H}_1}{\partial s(t)} = -\pi(t)g'(s(t)) = \begin{cases} -2\pi(t)(1-s(t)) & \text{logistic growth} \\ -\pi(t) & \text{exponential growth.} \end{cases} \tag{19b}$$

Next, the terminal time t_1 and the endpoint $s(t_1)$ of the travelling problem are free and may be chosen in an optimal way. While the arrival time t_1 determines the length of the harvesting period Δ, the endpoint $s(t_1) = s_1$ determines the initial value of the growth process for the harvesting problem. Together, both effects determine the maximal value $J_2^*(s_1, t_1)$ of the harvesting period, which in turn represents the scrap value of the compound problem (3). However, the arrival time t_1 coincidentally also determines the endpoint $s_1 = s(t_1)$, and for this reason we do not have two, but only one transversality condition representing both effects: the direct effect of the arrival time on the length of the harvesting period Δ, and the effect of t_1 on the stock at the beginning of that period $s(t_1)$.

To derive a necessary condition for the optimal choice of the arrival time t_1, we first have to substitute the transversality condition $s_1 = s(t_1) = s_0 e^{t_1}$, into J_2^*, to obtain, with slight abuse of notation, $J_2^*(t_1) \equiv J_2^*(s(t_1), t_1)$. Then, defining $V(t_1) \equiv -J_1^*(t_1) + J_{2A}^*(t_1)$ and using (18), the associated necessary condition for the free terminal time of the travelling problem t_1 reads as[25]

$$\frac{\mathrm{d}V(t_1^*)}{\mathrm{d}t_1} \equiv -\frac{\mathrm{d}J_1^*(t_1^*)}{\mathrm{d}t_1} + \frac{\mathrm{d}J_2^*(t_1^*)}{\mathrm{d}t_1} \equiv \mathcal{H}_1^*(t_1^*) + \frac{\mathrm{d}J_2^*(t_1^*)}{\mathrm{d}t_1} = 0. \tag{20}$$

With the help of condition (20) we are now able to calculate the optimal travelling–and–harvesting policy. We do this for both growth functions specified above.

[25]Condition (20) represents a modification of the usual necessary condition for the free terminal time, as provided, for example, by Léonard and Long (1992, Theorem 7.6.1).

5.3. Optimal travelling–and–harvesting policy for exponential growth. Acknowledging the transversality conditions, the following conditions have to be added to the canonical system:

$$s(t) = s_0 e^t, \qquad \pi(t) = M \left(\frac{1}{\bar{h}}\right)^{\frac{1}{\bar{h}-1}} e^{T-t}. \tag{21}$$

We next show that t_1 must not be smaller than the switching time τ, so that harvesting begins immediately upon arrival. Intuitively, this is because a premature arrival is costly without yielding any additional profit, as we initially have $h(t) = 0$ in Case B. Thus, a policy that implies Case B is never optimal, so that Case A must apply for the optimal policy; correspondingly, the maximised value function of the harvesting problem is given by eq. (6b).

Proposition 7. *In the optimal travelling–and–harvesting policy, the arrival takes place after the switching time τ,* i. e. *$t_1^* > \tau$, and thus Case A applies. Then, the optimal harvesting policy is characterised by Proposition 1, while the optimal travelling policy is given by Proposition 6. Hence, the resulting profit from the optimal travelling–and–harvesting policy is given by*

$$V(t_1^*) \equiv J_{2A}^*(t_1^*) - J_1^*(t_1^*) = s_0 M e^{t_1^*} \frac{\bar{h}}{\bar{h}-1} \left(1 - e^{(\bar{h}-1)(t_1^*-T)}\right) - \left(\frac{12x_1^2}{(t_1^*)^3} + cx_1\right), \tag{22a}$$

where the optimal arrival time t_1^* is a function of $\bar{h}$ and T, implicitly defined by

$$\frac{\mathrm{d}V(t_1^*)}{\mathrm{d}t_1} = \frac{36x_1^2}{(t_1^*)^4} - s_0 M e^{t_1^*} \frac{\bar{h}}{\bar{h}-1} \left(\bar{h} e^{(\bar{h}-1)(t_1^*-T)} - 1\right) = 0. \tag{22b}$$

Proof. See Appendix B. □

Lemma 3. *The derivative of the maximised value function of the harvesting problem is determined by the switching point τ:*

$$\frac{\mathrm{d}J_{2A}^*(t_1)}{\mathrm{d}t_1} \lesseqgtr 0 \quad \Leftrightarrow \quad T - t_1 \gtreqless \delta \equiv \frac{\log(\bar{h})}{\bar{h}-1} \quad \Leftrightarrow \quad \tau \gtreqless t_1. \tag{23}$$

Proof. See Appendix B. □

Since δ is a decreasing function of $\bar{h}$, the derivative of J_{2A}^* is positive for large, and negative for small values of $\bar{h}$. Now, the condition $\mathrm{d}J_{2A}^*/\mathrm{d}t_1 = 0$ determines the optimal arrival time in the absence of any travelling cost. If the harvesting capacity, when compared with the length of the harvesting period $\Delta \equiv T - t_1$, is large, a given volume of harvest can be collected in a shorter time interval, thus giving scope for a later arrival. Since a later arrival, leaves the resource with more time to grow, a higher harvesting capacity allows the agent to postpone the arrival time. Conversely, when the harvesting capacity is relatively low, postponing the start of the harvesting activity

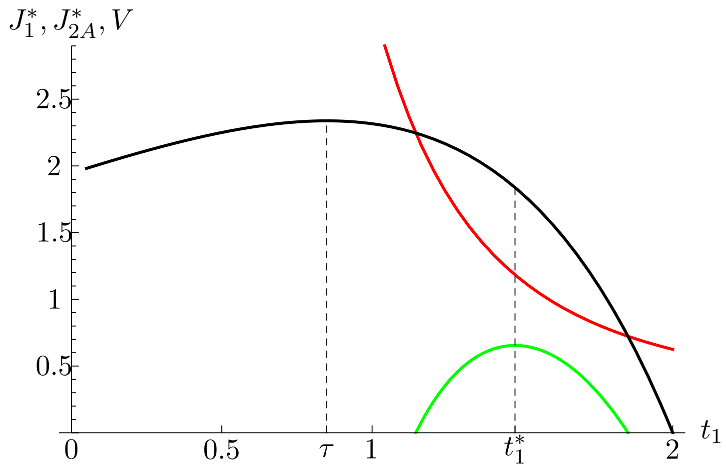

FIGURE 8. Exponential growth: revenue function J_{2A}^* (black), cost function J_1^* (red), and profit function V (green), for $M = 1$, $s_0 = 1$, $T = 2$, $c = 1/2$, $x_1 = 1/2$ and $\bar{h} = 3/4$, yielding switching time $\tau = 0.84927$, the optimal arrival time $t_1^* = 1.47525$ and the net profit $V(t_1^*) = 0.65429$.

is unattractive, as the agent will be unable to benefit from the higher stock due to the limited harvesting capacity. The optimal arrival time balances the benefits from an earlier and a later arrival:[26]

Corollary 1. *In the absence of any travelling cost, the optimal arrival time equals the switching time,* i. e. $t_1^\circ = \tau \equiv T - \delta$.

We thus find that the optimal arrival time t_1^* is chosen so that the harvesting activity begins immediately upon arrival (Case A). In other words, the optimal arrival time is relatively late in the sense $T - t_1^* < \delta$ or $t_1^* > \tau$, so that the agent begins with harvesting at the maximum rate immediately at time t_1. This is because an early arrival results in higher travelling costs and curtails the time for further growth. Both effects are unwelcome, so early arrival should be avoided. The determination of the optimal arrival time t_1^* and the switching time τ is illustrated in Figure 8.

It follows from Proposition 7 and Corollary 1 that higher travelling costs imply later arrival, and that the optimal arrival time t_1^* exceeds the optimal arrival time when travelling is costless t_1°, which in turn coincides with the switching time: $t_1^* > t_1^\circ = \tau \equiv T - \delta$. Conversely, the optimal length of the harvesting period $\Delta^* \equiv T - t_1^*$ (assuming $T > \Delta^*$) is smaller than the harvesting period the agent would have chosen in the absence of any travelling cost $\Delta^\circ \equiv T - t_1^\circ$.

5.4. Optimal travelling–and–harvesting policy for logistic growth. In the case of logistic growth we assumed that the resource was left unimpaired for a sufficiently

[26]Indeed, Case A and Case B coincide for $\tau = t_1$.

long time, so that upon arrival the stock rests at its steady state level $s(t_1) = s_1 = s^*$. Consequently, in this case the arrival time t_1 can be chosen without affecting s_1, but still we have to consider two cases: Case A and Case B.

5.4.1. *Case A: either $\bar{h} < 1$ or $1 < \bar{h} < 2$ and $T \leq T_c$.* In this case, the optimal harvesting policy is characterised by Proposition 4; and the optimal travelling policy by Proposition 6. Hence, the functions $J^*_{2A}(t_1)$ and $J^*_1(t_1)$ are given by eqs (12b) and (17), respectively. Moreover, the derivative of $J^*_{2A}(t_1)$ equals

$$\frac{\mathrm{d}J^*_{2A}(t_1)}{\mathrm{d}t_1} = -\frac{2\left(\bar{h}-2\right)\bar{h}M}{\bar{h}e^{\left(\bar{h}-2\right)(T-t_1)}-2} \leq 0, \tag{24}$$

which is negative for all $0 < \bar{h} \leq 2$, as the numerator and the denominator are both negative, and $\lim_{\bar{h}\nearrow 2} \mathrm{d}J^*_{2A}(t_1)/\mathrm{d}t_1 = -4M/\left(2(T-t_1)+1\right) < 0$.[27] Since $\mathrm{d}J^*_1(t_1)/\mathrm{d}t_1$ is also negative, the optimal arrival time is determined by eq. (20), and the optimal travelling–and–harvesting policy is characterised as follows:

Proposition 8. *Let either $\bar{h} < 1$ or $1 < \bar{h} < 2$ and $T \leq T_c$. Then, the optimal harvesting policy is $h(t) = \bar{h}$ for all $t \in \Delta$, and the resulting profit from the optimal travelling–and–harvesting policy, $V(t^*_1) \equiv J^*_{2A}(t^*_1) - J^*_1(t^*_1)$, is given by*

$$V(t^*_1) = \bar{h}M \log\left(\frac{2e^{(2-\bar{h})(T-t^*_1)}-\bar{h}}{2-\bar{h}}\right) - \left(cx_1 + \frac{12x_1^2}{(t^*_1)^3}\right), \tag{25a}$$

where t^*_1 is a function of $\bar{h}$ and T, implicitly defined by

$$\frac{\mathrm{d}V(t^*_1)}{\mathrm{d}t_1} = \frac{36}{(t^*_1)^4} - \frac{2\left(\bar{h}-2\right)\bar{h}M}{\bar{h}e^{\left(\bar{h}-2\right)(T-t^*_1)}-2} = 0. \tag{25b}$$

Proof. The result follows from the preceding analysis. □

The functions $J^*_1(t_1)$ and $J^*_{2A}(t_1)$ are depicted in Figure 9 for a low (left diagram) and a high (right diagram) harvesting capacity, with $T = 5$ and $M = 1$. Setting $\bar{h} = 3/4$, the optimal arrival time equals $t^*_1 = 2.4793$ yielding a net profit equal to $V(t^*_1) \equiv J^*_{2A}(t^*_1) - J^*_1(t^*_1) = 1.8161$. Observe that Case A actually materialises for $T = 5$ with $M = 1$ and $c = 1/10$. (Compare the second diagram in Figure 6.)

5.4.2. *Case B:* $1 < \bar{h} < 2$ *and* $T > T_c$. In this case, the harvesting capacity $\bar{h}$ exceeds the critical value $\bar{h}_c$. Then, the optimal harvesting policy is characterised by Proposition 5, and the associated profit is given by eq. (14b). Accordingly, the derivative of the value function equals $\mathrm{d}J^*_{2B}(t_1)/\mathrm{d}t_1 = -M$, and therefore the optimal arrival time t^*_1 can be calculated explicitly:

[27] Moreover $\mathrm{d}J^*_{2A}(t_1)/\mathrm{d}t_1 = 0$ for $\bar{h} = 0$.

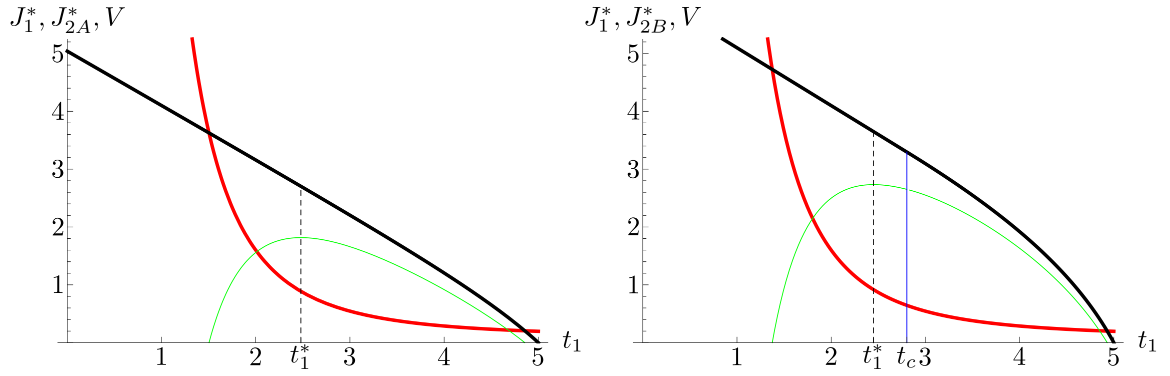

FIGURE 9. Logistic growth: value function J_2^* (black), cost function J_1^* (red), and profit function V (green), for $M = 1$, $T = 5$ and $c = 1/10$. Case A (left): $\bar{h} = 3/4$, optimal arrival time $t_1^* = 2.4793$. Case B (right): $\bar{h} = 3/2$, optimal arrival time $t_1^* = \sqrt{6} = 2.4495$ with critical arrival time $t_c = 5 - 2\log(3) = 2.8028$ (blue).

Proposition 9. *Let $1 < \bar{h} < 2$ and $T > T_c$. Then, the optimal harvesting policy is characterised by Proposition 5, and the resulting profit from the optimal travelling–and–harvesting policy, $V(t_1^*) \equiv J_{2B}(t_1^*) - J_1(t_1^*)$, is given by*

$$V(t_1^*) = M\left[T - t_1^* + 2\bar{h}\log\left(\frac{\bar{h}}{\bar{h}-1}\right) - \frac{1}{2-\bar{h}}\log\left(\frac{\bar{h}}{2(\bar{h}-1)^2}\right)\right] - \left(cx_1 + \frac{12x_1^2}{(t_1^*)^3}\right), \quad (26)$$

with $t_1^* = \sqrt{6x_1/\sqrt{M}}$.

Proof. The result follows from the preceding analysis. □

This scenario is depicted for $\bar{h} = 3/2$ (and $T = 5$, $c = 1/10$, $M = 1$) in the right panel of Figure 9, where the value function $J_{2B}^*(t_1)$ has slope -1 for all arrival times $t_1 < t_c$. (Recall that for these parameter values, Case B results, which is illustrated in the second diagram in Figure 6). With these parameters, the optimal solution is given by $t_1^* = 2.4495$ yielding a net profit of $J_{2B}(t_1^*) - J_1(t_1^*) = 2.7326$.[28]

Finally, the optimal arrival time for the case that travelling is costless is the same for both Case A and Case B:

Corollary 2. *In the absence of travelling costs, the optimal arrival time is equal to the earliest (physically) feasible arrival time,* i. e. $t_1^\circ = t_{min}$.

Proof. The result follows from the fact that in both Case A and Case B, the derivative of J_2 is negative for any $h > 0$, and hence the optimal arrival time t_1° is chosen minimally. □

[28]With a later starting time $t_1 > t_c \equiv T - T_c = 5 - 2\log(3) = 2.8028$, Case A would become relevant, as the time left is less than the minimal time interval required for harvesting in Case B, $T_c = 2\log(3) = 2.1972$.

6. Conclusion

In this paper we contribute to the theory of spatial resource economics by explicitly taking into account the fact that in many real-world situations, the agent has to travel to the location of the resource before being able to harvest. Although some papers acknowledge the requirement of an agent to travel to the resource (*e. g.* Behringer and Upmann, 2014; Belyakov et al., 2015), the approach of this paper differs in that the resource cannot be harvested in an *en passant* manner, but the agent has to stop at the location of the resource in order to harvest. As a consequence, travelling and harvesting are two mutually exclusive activities, yet the travelling problem and the subsequent harvesting problem are closely linked: since the speed of travelling and thus resulting arrival time determines both, the start of the harvesting period and the initial value of the resource stock, the arrival time becomes the crucial decision for the optimal harvesting policy.

We are able to fully characterise the control programme for this combined travelling–and–harvesting problem, employing recent tools from two-stage dynamic optimization theory. To this end, we analyse the two sub-problems in reverse order: we first solve the harvesting problem (Proposition 1–5), and then the preceding travelling problem (Proposition 6) taking into account the optimal harvesting policy of the second stage. This allows us to highlight the critical linkage between both stages and thus to characterise the resulting optimal yield for the management of the remote resource. We derive the optimal policy for two different growth processes of the resource stock: exponential growth (Proposition 7) and logistic growth (Propositions 8 and 9).

Further robustness checks (Appendix A) involve the case of a positive discount rate and bounds on acceleration. We show that a positive discount rate lets the agent postpone part of the travelling cost by shifting the acceleration profile and hence the speed cost towards the future; bounds on acceleration, though, hamper this cost–shifting effect as, in the case of a binding lower bound, the agent is required to slow down earlier, and hence the initial acceleration must be higher.

In sum, we have demonstrated that acknowledging the spatial dimension in the classical problem of managing a renewable resource can lead to interesting and economically relevant, yet analytically tractable changes. Our set-up allows for the introduction of further realistic features such as separate periods of travelling and harvesting. Besides following the call for introducing a spatial dimension and thus enhancing the realism of harvesting models, our model allows for an extension of the theory to the case where the agent faces a transportation problem that is temporarily and spatially linked to the resource–gathering problem.

Appendix A. Robustness analysis: a positive discount rate and bounds on acceleration

We here explore how a positive discount rate and bounds on acceleration affect the optimal travelling policy. We maintain our specification of travelling cost (15). Then, with a positive discount rate, the objective function thus equals[29]

$$J_1(a, t_1) = \int_0^{t_1} e^{-\rho t} \left(cv(t) + a(t)^2 \right) \, \mathrm{d}t.$$

Acknowledging the law of motion (1a), the Hamiltonian is given by

$$\mathcal{H}_1 = -cv(t) - a(t)^2 + \psi_1(t)v(t) + \psi_2(t)a(t).$$

We assume that acceleration is bounded to $\mathscr{A} = [\underline{a}, \overline{a}] = [-1, +1]$, and the corresponding Lagrangean reflecting this constraint equals

$$\mathcal{L} = \mathcal{H}_1 + \lambda_1(t)(1 + a(t)) + \lambda_2(t)(1 - a(t)).$$

Then, the necessary conditions for the optimal solution are given by

$$a(t) = \frac{1}{2} \left(\pi_2(t) + \lambda_1(t) - \lambda_2(t) \right), \tag{A.1a}$$

$$\dot{x}(t) = v(t), \quad x(0) = 0, \quad x(t_1) = x_1, \tag{A.1b}$$

$$\dot{v}(t) = a(t), \quad v(0) = 0, \quad v(t_1) = 0, \tag{A.1c}$$

$$\dot{\pi}_1(t) = -\frac{\partial \mathcal{L}}{\partial x(t)} + \rho \pi_1(t) = \rho \pi_1(t), \tag{A.1d}$$

$$\dot{\pi}_2(t) = -\frac{\partial \mathcal{L}}{\partial v(t)} + \rho \pi_2(t) = -\pi_1(t) + \rho \pi_2(t) + c. \tag{A.1e}$$

Using the yet unspecified initial values $\pi_1(0) = m_1$ and $\pi_2(0) = c_1$, the latter two equations yield

$$\pi_1(t) = m_1 e^{\rho t}, \qquad \pi_2(t) = e^{\rho t} \left(c_1 - m_1 t \right) + \frac{c}{\rho} \left(e^{\rho t} - 1 \right). \tag{A.2}$$

To be able to clearly identify the effects of the bounds on the controls, we first solve, as a point of reference, the problem when a is unbounded.

A.1. Analysis of the unbounded solution. In this case we have $\lambda_1(t) = 0 = \lambda_2(t)$, so that (A.1a) reduces to $a(t) = \pi_2(t)/2$. Substituting this into (A.1b) and (A.1c), we obtain

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = \frac{\pi_2(t)}{2}. \tag{A.3}$$

[29] In order to avoid that a negative speed, *i. e.* moving backwards, reduces cost, it is preferable to write $c|v|$ instead of cv. However, we will assume that c is sufficiently low so that moving backwards is never optimal and both cost components are non-negative along the optimal path. Still, for sufficiently high values of c, moving backwards may be part of the solution unless we replace cv by $c|v|$.

Then, substituting eq. (A.2) into (A.3) yields the system

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = \frac{1}{2}\left(e^{\rho t}\left(c_1 - m_1 t\right) + \frac{c}{\rho}\left(e^{\rho t} - 1\right)\right), \quad x(0) = 0, \quad v(0) = 0,$$

the solution of which is given by

$$x(t) = \frac{1}{4\rho^3}\left[c\left(2e^{\rho t} - 2 - \rho t(\rho t + 2)\right) - 2c_1\rho\left(\rho t - e^{\rho t} + 1\right)\right.$$
$$\left. -2m_1\left(\rho t + e^{\rho t}(\rho t - 2) + 2\right)\right], \tag{A.4a}$$
$$v(t) = \frac{1}{2\rho^2}\left[c\left(e^{\rho t} - 1 - \rho t\right) + e^{\rho t}\left(c_1\rho + m_1 - m_1\rho t\right) - c_1\rho - m_1\right]. \tag{A.4b}$$

Finally, using the terminal conditions $x(t_1) = x_1$ and $v(t_1) = 0$, we obtain the constants c_1 and m_1. Applying those in (A.2), and using that result to calculate $a(t) = \pi_2(t)/2$, we obtain a generalisation of Proposition 6, with the limiting case $\rho \to 0$ recouping our former result:

Proposition 10. *Given arrival time t_1, the optimal travelling policy is given by*

$$a(t) = \frac{1}{4\rho}\frac{\theta_1(t, t_1)}{\theta_3(t_1)}\, c - \frac{\theta_2(t, t_1)}{\theta_3(t_1)}\, x_1, \tag{A.5a}$$

and the associated travelling cost amounts to

$$J_1^*(t_1) = -\frac{1}{\kappa_0(t_1)}\left(\kappa_1(t_1, x_1) + \kappa_2(t_1, x_1)\, c + \kappa_3(t_1)\, c^2\right), \tag{A.5b}$$

where

$$\begin{aligned}
\theta_1(t, t_1) &\equiv 2 + 2e^{2\rho t_1} - 2e^{\rho t_1}\left(\rho^2 t_1^2 + 2\right) + \rho t_1 e^{\rho t}\left(\rho t(2 + \rho t_1) + \rho t_1 + 4\right) \\
&\quad - \rho t_1 e^{\rho(t+t_1)}\left(\rho t(2 - \rho t_1) + \rho t_1(\rho t_1 - 3) + 4\right), \\
\theta_2(t, t_1) &\equiv \rho^2 e^{\rho t}\left(1 + \rho t + e^{\rho t_1}\left(\rho(t_1 - t) - 1\right)\right), \\
\theta_3(t_1) &\equiv e^{\rho t_1}\left(\rho^2 t_1^2 + 2\right) - e^{2\rho t_1} - 1, \\
\kappa_0(t_1) &\equiv 2 + \rho^2 t_1^2 - 2\cosh(\rho t_1), \\
\kappa_1(t_1, x_1) &\equiv \rho^3 x_1^2 e^{-\rho t_1}\left(e^{\rho t_1} - 1\right), \\
\kappa_2(t_1, x_1) &\equiv 8\rho^4 t_1 x_1\left(\rho t_1 \cosh\left(\rho t_1/2\right) - 2\sinh\left(\rho t_1/2\right)\right), \\
\kappa_3(t_1) &\equiv \sinh\left(\rho t_1/2\right)\left(\rho^4 t_1^4 + 12\rho^2 t_1^2 - 8\cosh(\rho t_1) + 8\right) - 4\rho^3 t_1^3 \cosh\left(\rho t_1/2\right).
\end{aligned}$$

Proof. The result follows from the preceding analysis. □

We thus find that for a fixed arrival time t_1, the optimal control a is an affine function of c (as are x, v and π_2, while π_1 is independent of c); and for this reason, the minimised travelling cost J_1^* is a polynomial of order 2 of c.

A.2. Analysis of the bounded solution. We now take into account that the control is actually bounded to $a(t) \in \mathscr{A}$. The optimal control of the unbounded case may either hit the upper bound or the lower bound, or both. For illustrative purposes, let us henceforth assume that the parameters are chosen such that the lower bound becomes binding, while the upper bound does not. In this case, it suffices to consider the constraint $a \geq \underline{a}$ only. (The case $a \leq \overline{a}$ can be analysed in an analogous way.)

Assume that for the unbounded solution, the lower bound becomes binding at time $\underline{t} \in (0, t_1)$. We refer to $\underline{t}$ as the *hitting time*. The *optimal* hitting time for the bounded case ξ, though, has to be determined as part of the solution of the bounded problem. We know that we have $a(t) = \underline{a} = -1$ for all t in the final interval $(\xi, t_1]$. Consequently, we must have $\xi < \underline{t}$, for if $\xi = \underline{t}$ the remaining time would only suffice to guarantee the terminal condition $v(t_1) = 0$, if we were able to set $a < \underline{a}$. Thus, during the final time interval $(\xi, t_1]$, the solution must satisfy

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = -1, \quad x(t_1) = x_1, \quad v(t_1) = 0,$$

the solution of which is

$$x(t) = \frac{1}{2}\left(-(t - t_1)^2 + 2x_1\right), \quad v(t) = t_1 - t.$$

In particular, these equations must be satisfied at the optimal hitting time ξ, and thus they provide the right-hand boundary conditions for the time interval $[0, \xi]$. Hence, we obtain a system consisting of (A.1d), (A.1e), (A.3) and the boundary conditions:

$$x(0) = 0, \quad v(0) = 0, \quad x(\xi) = \frac{1}{2}\left(-\xi^2 + 2\xi t_1 - t_1^2 + 2x_1\right), \quad v(\xi) = t_1 - \xi.$$

The solution of this system yields the optimal travelling policy for the interval $[0, \xi]$, which is again an affine function of c. Hence, putting the pieces together, we find:

Proposition 11. *Given arrival time t_1 and the constraint $a \in \mathscr{A} \equiv [-1, 1]$, the optimal travelling policy is given by*

$$a(t, \xi) = \begin{cases} \dfrac{1}{4\rho}\dfrac{\theta_1(t, \xi)}{\theta_3(\xi)}c + \dfrac{\rho}{2}\dfrac{\theta_5(t, \xi)}{\theta_3(\xi)}, & \forall t \in [0, \xi] \\ -1, & \forall t \in (\xi, t_1] \end{cases}$$

where the optimal hitting time ξ is determined by $a(\xi, \xi) = -1$, θ_1 and θ_3 are as defined in Proposition 10, and

$$\begin{aligned} \theta_5(t, \xi) \equiv\, & e^{\rho t}\left[-4\xi + \rho^2 t\left(-\xi^2 + t_1^2 - 2x_1\right) + \rho(t_1 - \xi)(\xi + 2t + t_1) + 4t_1 - 2\rho x_1\right] \\ & + e^{\rho(\xi + t)}\left[4\xi - \rho(t_1 - \xi)(-3\xi + 2t + t_1)\right. \\ & \qquad \left. + \rho^2(\xi - t)\left((t_1 - \xi)^2 - 2x_1\right) - 4t_1 + 2\rho x_1\right]. \end{aligned}$$

Proof. The result follows from the preceding analysis. ☐

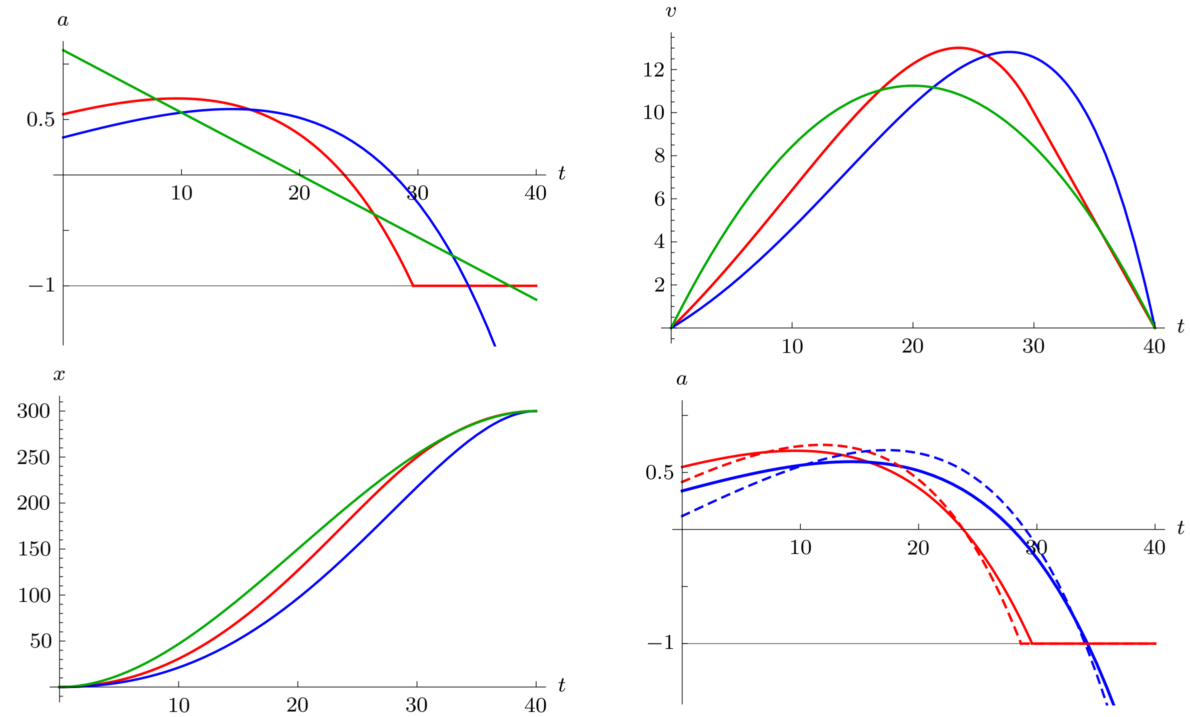

FIGURE 10. Optimal acceleration (upper left), speed (upper right) and position (lower left), with (red) and without (blue) bounds on acceleration for $\rho = 1/20$; and, for comparison, without bounds for $\rho = 0$ (green). Lower right: optimal acceleration for $c = 1/10$ (bold curves) and $c = 1/5$ (dashed curves) for $\rho = 1/20$.

Example. In order to illustrate our results, we apply the parameter specification: $t_1 = 40, x_1 = 300, \rho = 1/20, c = 1/10$. For this specification, the unbounded solution hits the lower bound at time $\underline{t} \approx 34.2818$ (see blue case in Figure 10); while the resulting optimal hitting time equals $\xi = 29.5984$. Combining the optimal control before and after the hitting time, we obtain

$$a(t) = \begin{cases} \frac{1}{2}\left(e^{t/20}(3.09497 - 0.104565\, t) - 2\right) & 0 \leq t \leq \xi = 29.5984, \\ -1 & \xi < t \leq t_1 = 40, \end{cases}$$

with minimal travelling cost amounting to $\overline{J}_1^* = 18.4648$. The optimal solution is illustrated by the red trajectories in Figure 10. As expected, compared to $J_1^* \approx 16.7095$ for the case of an unbounded control, the presence of the bound on acceleration results in an increase in the minimal travelling cost.

We may also compare our result for a positive discount rate with the case $\rho = 0$, displayed by the green trajectories in Figure 10. In the absence of discounting, we obtain $J_1^*|_{\rho=0} = 375/8 = 46.875$, so that discounting makes part of the cost vaporize. This can also be seen in Figure 10 by comparing the blue with the green paths: a positive discount rate makes the agent initially move more slowly and speed up later so that part of the travelling cost is shifted to the future. (In case of bounds on the control, such cost shifting becomes limited so that some part of the travelling cost must be incurred earlier, compare the red paths.) This cost–shifting effect is the more pronounced the higher the cost of speed is: compare the optimal acceleration path for $c = 1/10$ (bold curves) with that for $c = 1/5$ (dashed curves) in the lower right panel of Figure 10. $\Diamond$

The presence of both cost–shifting effects can formally be shown:

Corollary 3. *Given arrival time t_1 and the constraint $a \in \mathscr{A} \equiv [-1, 1]$, and the associated optimal travelling policy, we have*

(i) $\partial J_1^*/\partial\rho < 0$ *for all ρ and $c \leq \bar{c}(\rho)$, where $\bar{c}(\rho)$ is the highest cost parameter such that for any $c > \bar{c}(\rho)$ the agent begins the travelling period with moving backwards,* i. e. $a(0)|_{c=\bar{c}(\rho)} = 0$, *(see also fn. 29);*
(ii) $\partial a(0, \xi)/\partial c < 0$, i. e. *the acceleration at the beginning of the period falls if the cost of speed increases.*

Proof. (i): The result follows from the Envelope Theorem. (ii): Using Proposition 11, and evaluating the optimal control at $t = 0$, we obtain (with slight abuse of notation):

$$\frac{\partial a(0,\xi)}{\partial c} = \frac{1}{2\rho}\left(-1 + \gamma(\rho t_1)\right) \quad \text{where} \quad \gamma(z) \equiv \frac{z\left(e^z z^2 - 3e^z z - z + 4e^z - 4\right)}{2\left(-e^z z^2 - 2e^z + e^{2z} + 1\right)}.$$

Using l'Hôpital's rule, we find $\lim_{z\to 0}\gamma(z) = 1$ and $\lim_{z\to\infty}\gamma(z) = 0$. Moreover, it is tedious but straightforward to show that $\gamma' < 0$, and hence we have $\partial a(0, \xi)/\partial c < 0$. $\square$

Appendix B. Proofs

Proof of Lemma 1.

Proof. Assume, on the contrary, that $\pi(t_1) > M$. Since $s(t_1) = 2$, it follows from eq. (8c) that $\dot{\pi}(t_1) = -h(t_1)M + \pi(t_1)\left(2 + h(t_1)\right) > 0$. Since $h(t) = 0$ as long as $\pi(t) > M$, the stock remains at its starting value $s(t_1) = 2$. Given this, there is no turning point in the evolution of π, and thus π continues to grow, *i. e.* we have $\dot{\pi}(t) > 0$ for all t. Yet, this contradicts the transversality condition $\pi(T) = 0$ and thus proves our claim $\pi(t_1) < M$, and thus $h(t_1) = \bar{h}$. $\square$

Proof of Lemma 2.

Proof. From (12a) we can calculate the critical time horizon T_c for which at some point in time t_c the trajectory goes through the point $(s(t_c), \pi(t_c)) = (1, M)$. Using that information and evaluating π at T_c yields $t_c = t_1 + \frac{1}{2-\bar{h}}\log\left(\frac{\bar{h}}{2(\bar{h}-1)}\right)$ and thus eq. (11). $\square$

Proof of Lemma 3.

Proof. The derivative of the value function $J_{2A}^*(t_1)$ is given by:

$$\frac{\mathrm{d}J_{2A}^*(t_1)}{\mathrm{d}t_1} = -s_0 M e^{t_1} \frac{\bar{h}}{\bar{h}-1}\left(\bar{h}e^{(\bar{h}-1)(t_1-T)} - 1\right).$$

It is then straightforward to show that, irrespective of the sign of $\bar{h}-1$, the sign of the derivative of J_{2A}^* depends on whether the switching point τ is before or after the arrival time t_1, and hence (23) follows. □

Proof of Proposition 3.

Proof. (A similar proof can be found in Hocking, 1991.) Since the Hamiltonian is autonomous, it is constant along the optimal trajectory (see, for example, Intriligator, 1971, p. 350, 355). We can therefore characterise the trajectories in the (s, π) plane for $h = 0$ and for $h = \bar{h}$. Let K denote the level of the Hamiltonian, then the optimal trajectories are characterised by the equations

$$\pi(t) = \frac{K}{2s(t) - s^2(t)} \quad \text{and} \quad \pi(t) = \frac{K - s(t)\bar{h}}{2s(t) - s^2(t) - s(t)\bar{h}}$$

for $h = 0$ and $h = \bar{h}$, respectively. The $h = 0$ trajectories have their minima at $s = 1$, and the trajectories with $h = \bar{h}$ attain their maxima along the curve

$$\pi(t) = \frac{-\bar{h}}{2 - 2s(t) - \bar{h}} \quad \text{for } s > 1 - \frac{1}{2}\bar{h}.$$

Both types of trajectories are depicted in Figure 5 for a low (left diagram) and a high (right diagram) harvesting capacity. The trajectories starting from $s(t_1) = 2$ reach the horizontal axis at time T, *i. e.* $\pi(T) = 0$. Those trajectories with $\bar{h} < 1$ cross the horizontal axis at a point to the right of $2 - \bar{h}$, that is $s(T) > 2 - \bar{h}$. If $\bar{h}$ is sufficiently small, the trajectory does not reach the $\pi = M$ line (for $M = 1$ see Figure 5, left). Since the locus of maxima crosses the point $(1, M)$, the critical trajectory is that one which achieves its maximum at this point (see Figure 5, right). Because the trajectories do not cross the horizontal axis to the left of $2 - \bar{h}$, the critical trajectory must feature $\bar{h} > 1$. It thus follows that the critical harvesting capacity exceeds unity, $\bar{h}_c > 1$. □

Proof of Proposition 5.

Proof. That equation (14a) is indeed the optimal harvesting policy can be seen by noting that $\pi = M$ is a singular level. Since we already know that $\pi(t_1) < M$, it follows that $\dot{\pi} \geq 0$ at the time the singular level $\pi = M$ is reached. If we have $\pi = M$ for some time interval with positive length, then $\dot{\pi} = 0$ and hence we must have $s = 1 \Rightarrow \dot{s} = 0 \Rightarrow h = 1$ from eq. (8a).

After completing the singular path we cannot have a path with $h = 0$. This can be seen as follows: $h = 0$ implies $\dot{s} > 0$, which in turn implies that, because $s = 1$ on the singular arc, $s > 1$ right after the singular arc. Jointly with $h = 0$ this in turn implies that $\dot{\pi} > 0$. Hence, we enter a path where both s and π are growing so that the transversality condition $\pi(T) = 0$ cannot be satisfied. We thus conclude that the optimal policy must proceed with $h = \bar{h}$ after completing the singular path—and thus the policy in eq. (14a) is optimal.

The total length of the harvesting sub-periods $[t_1, t_2)$ and $[t_3, T]$ amounts to $T_c - t_1$ and is thus given by eq. (11). Therefore harvesting during these sub-periods brings about the same profit as in Case A, *i. e.* J^c_{2A} given by eq. (13); while during the time interval $[t_2, t_3)$ the resulting profit equals $h = 1$ times the length of the harvesting period: $t_3 - t_2 = T - T_c$. So, we obtain $J^*_{2B} = J^c_{2A} + M \int_{t_2}^{t_3} 1 \, \mathrm{d}t = J^c_{2A} + M\,(T - T_c)$. Finally, substituting the definition of T_c, eq. (11), yields eq. (14b). □

Proof of Proposition 7.

Proof. Using the maximised Hamiltonian of sub-section 5.1, $\mathcal{H}^*_1 = 36x_1^2/t_1^4$, the transversality condition (20) gives eq. (22b). Since $\mathcal{H}^*_1 = -\mathrm{d}J^*_1/\mathrm{d}t_1 > 0$, the derivative $\mathrm{d}J^*_{2A}(t_1)/\mathrm{d}t_1$ must be negative in order for (22b) to have a solution t^*_1. By Lemma 3, $\mathrm{d}J^*_{2A}(t_1)/\mathrm{d}t_1$ is negative if, and only if, the switching point τ is before the arrival time: $t^*_1 > T - \delta \equiv \tau$, implying that Case A applies. □

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